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CONTENTS.

PART I.

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PREFACE.

The rapid growth of the aircraft industry has led to the recruitment of large numbers of draughtsmen from other branches of engineering, many of whom have little real knowledge of the principles of aircraft stressing. This book is written especially for these men, and also for aircraft draughtsmen generally; for such knowledge is indispensable to the draughtsman-designer or section-leader, and only slightly less so to the detail draughtsman.

Principles alone are the chief concern; it would not be possible to compress into a book of this nature a complete treatise on the subject of aircraft stressing, but by using a fair amount of common sense, upon which, combined with some imagination, successful stressing is largely dependent, the reader should be able to attack most problems in the course of his everyday work.

Regarding the rather difficult question of what knowledge to assume in the reader at the outset, the writer feels that familiarity with mechanics, dynamics, elementary strength of materials, and some knowledge of the calculus and of the functions of the individual units of the modern aeroplane must be presupposed, for, even if these matters could be adequately dealt with here, there would be no point in reiterating matter so thoroughly treated in standard text-books. Wherever a certain amount of recapitulation has been thought advisable, however, it has been kept down to an absolute minimum so as to allow for a concentrated attack on stressing proper.

The method of approach is largely by means of worked examples, based on actual practice, supplemented by what should be ample descriptive matter

The greater part of the subject-matter of the book has appeared as a series in *Practical Engineering*, and the author is indebted to Messrs Newnes, the publishers of that weekly, for permission to reprint the articles in book-form.

He would also like to thank Messrs General Aircraft Ltd. for the use of certain of the curves and other data, and his colleagues, C. W. Prower, B.Sc , A.F.R.Ae.S., and H. M. J. Kittelsen, B.Sc., D I.C., for their many helpful suggestions.

W. L. M.

KINGSTON-ON-THAMES,
September, 1941.

THE PRINCIPLES OF AIRCRAFT STRESSING.

PART I.

CHAPTER I.

MOMENTS OF INERTIA—MODULUS OF SECTION.

WHEN a member is subjected to bending (i.e., when the external loading is such as to cause a bending moment), it is necessary to check its strength to make sure that the bending stress developed does not exceed the allowable value for the material.

To do this, the Moment of Inertia (I), or the Modulus (Z), and the Bending Moment (M) at the section considered are first found. The Bending Stress (f) is then equal to $\frac{M}{Z}$ or $\frac{My}{I}$, where y is the distance from the Neutral Axis to the outermost fibre.

Modulus of Section.—If the cross-section of a fitting, spar, tube, etc. is of standard symmetrical form, I and Z are easily found from the formulæ in Tables I and II, but if unsymmetrical or of built-up form, it is necessary to calculate these values, a convenient method of tabulation being shown in the worked example (Example 1) below (page 4).

In this connection, a very useful rule to remember is the Parallel Axis Theorem, by which, given I about the Neutral Axis (N.A.), I about any axis parallel to the N.A. can be found, and vice versa.

Thus, if A = area of section (in.²),

I_{NA} = moment of inertia about N.A. (in.⁴),

I_{CC} = required moment of inertia about axis CC (in.⁴), and

h = distance between N.A. and CC (in.), then

$$I_{CC} = I_{NA} + Ah^2.$$

Table I.—Values of A, I and Z for Standard Symmetrical Sections.

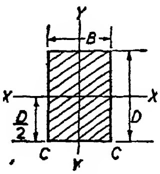
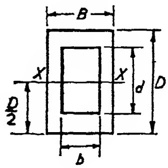
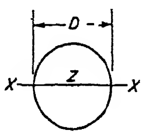
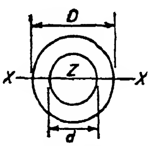
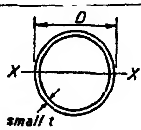
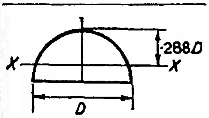
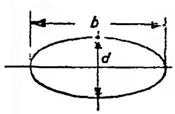
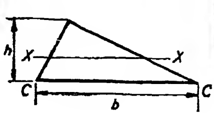
Section.		Area (A) (in.^{2}).	Axis.	Moment of Inertia (I) (in.^{4}).	Section Modulus (Z) (in.^{3}).
Rectangle		BD	XX	$\frac{BD^3}{12}=\frac{AD^2}{12}$	$\frac{BD^2}{6}$
			YY	$\frac{DB^3}{12}=\frac{AB^2}{12}$	$\frac{DB^2}{6}$
			CC	$\frac{BD^3}{3}=\frac{AD^2}{3}$	—
Hollow Rectangle		$(BD-bd)$	XX	$\frac{1}{12}(BD^3-bd^3)$	$\frac{1}{6}\frac{(BD^3-bd^3)}{D}$
Circle		$\frac{\pi}{4}D^2$	XX or any diameter.	$\frac{\pi}{64}D^4$	$\frac{\pi}{32}D^3$
			Polar ZZ	$\frac{\pi D^4}{32}$	—
Hollow Circle		$\frac{\pi}{4}(D^2-d^2)$	XX or any diameter	$\frac{\pi}{64}(D^4-d^4)$	$\frac{\pi}{32}\frac{(D^4-d^4)}{D}$
			Polar ZZ	$\frac{\pi}{32}(D^4-d^4)$	—
Thin Circular Tube		πDt	XX or any diameter	$\frac{\pi}{8}D^3t$	$\frac{\pi}{4}D^2t$
Semicircle		$\frac{\pi}{8}D^2$	XX	$007D^4$	$\cdot 0243D^3$
Ellipse		$\frac{4}{\pi}bd$	Major axis XX	$\frac{\pi}{64}bd^3$	$\frac{\pi}{32}bd^2$
			Minor axis YY	$\frac{\pi}{64}db^3$	$\frac{\pi}{32}db^2$
Triangle		$\frac{1}{2}bh$	XX through N.A.	$\frac{bh^3}{36}$	$\frac{bh^2}{24}$
			Base CC	$\frac{bh^3}{12}$	—

Table II.—Values of k, A, I and Z for Hollow Circular Tubes.

Dia.		$\frac{3}{4}''$	$\frac{7}{8}''$	$1''$	$1\frac{1}{8}''$	$1\frac{1}{4}''$	$1\frac{3}{8}''$	$1\frac{1}{2}''$	$1\frac{5}{8}''$	$1\frac{3}{4}''$	$1\frac{7}{8}''$	$2''$	$2\frac{1}{8}''$	$2\frac{1}{4}''$	$2\frac{3}{8}''$	$2\frac{1}{2}''$	$2\frac{5}{8}''$	$2\frac{3}{4}''$	$2\frac{7}{8}''$	$3''$
		·750	875	1·00	1 125	1·250	1·375	1·50	1 625	1·750	1 875	2 00	2 125	2·250	2 375	2 500	2 625	2 750	2·875	3·00
24G. (022″)	K	·2575	3017	·3458	·3900	4342	4784	·5226	5668	·6109	·6552	6994	·7435	·7877						
	A	·0503	·0590	·0676	·0762	·0849	·0935	·1021	·1108	·1194	·1281	·1367	·1455	·1540						
	I	·0033	0054	0081	0116	0161	·0215	·0279	·0356	·0445	·0548	0667	0802	·0955						
	Z	·0089	·0123	·0162	0207	·0257	·0313	·0373	·0438	·0509	0585	·0667	·0755	·0849						
22G. (·028″)	K	·2555	2998	·3437	·3880	·4322	4762	·5205	·5647	·6089	·6531	6974	7414	7856	8296	·8740				
	A	0635	0745	0855	0965	·1075	·1185	·1295	·1405	·1515	·1625	·1735	·1845	·1955	·2065	·2175				
	I	·0042	·0067	·0101	0145	·0201	·0269	0351	·0448	·0562	0693	·0844	1015	·1207	·1423	·1661				
	Z	·0112	·0153	·0202	·0258	·0321	·0391	·0468	·0552	·0642	·0740	·0844	·0955	·1073	·1198	·1329				
20G. (·036″)	K	·2528	·2969	·3410	·3852	·4295	·4736	5176	·5621	6060	6502	6946	7384	·7828	8270	·8711	·9151	9596		
	A	·0807	0948	·1090	·1231	·1372	1514	·1655	·1797	·1938	2079	·2221	·2362	·2503	·2645	·2787	·2928	·3070		
	I	·0050	·0083	0127	·0183	0254	·0340	0442	0567	0711	0879	1071	1289	·1533	1807	2115	2456	·2827		
	Z	0135	0190	0254	0326	0407	·0495	0592	0698	·0813	0938	1071	·1213	·1363	·1522	·1692	·1871	·2056		
17G. (·056″)	K	2460	2900	3342	3784	·4226	4666	5109	·5551	5992	6434	6876	·7317	7759	8201	8643	·9081	9526	9967	1 0410
	A	1221	1441	1660	1880	2100	2320	2540	2760	2980	3200	3419	3639	3859	4079	4299	·4510	4739	4959	5178
	I	·0074	0121	0185	·0269	0376	·0506	0663	0851	·1070	·1324	1616	·1949	·2324	2743	3212	·3730	4301	4927	5611
	Z	0197	·0277	·0371	·0479	·0601	·0736	·0884	·1047	·1223	1412	1616	·1834	·2066	·2310	·2570	2842	3128	·3428	·3741
14G. (080″)	K	·2385	2821	·3265	3706	·4146	·4587	·5026	·5472	5910	6352	6792	·7235	7675	·8120	·8558	9003	9441	·9885	1 0327
	A	·1684	1998	2312	2636	·2941	·3255	·3569	3883	4197	4511	·4825	5139	·5444	·5768	·6082	·6396	6710	·7024	·7338
	I	0096	0160	0246	·0361	·0506	·0685	·0903	·1162	·1466	·1821	·2227	·2691	·3215	3802	4456	5180	5983	6863	7825
	Z	·0256	0365	0493	0641	0809	0997	1204	·1430	1676	1947	2227	·2533	2858	3202	3565	3918	4351	4774	5217
10G (128″)	K				·3554	·3992	·4432	·4872	·5318	·5752	6193	·6634	·7075	·7516	·7961	·8398	·8840	·9281	·9722	1·0164
	A				4009	4512	5014	5517	·6019	6522	7025	7527	·8030	8533	9035	·9538	1 0040	1 0543	1·1046	1 1518
	I				·0506	·0718	·0984	·1308	·1697	·2158	2694	·3313	·4019	·4821	·5720	·6726	·7843	9079	1·0439	1·1929
	Z				0900	1149	·1431	·1743	2089	·2466	·2874	·3313	·3783	·4285	·4817	·5381	·5976	·6603	·7262	·7953

k is the radius of gyration and is such that $I = Ak^2$

For instance, consider a rectangle whose moment of inertia about base CC (Fig. 1) is required. Knowing that

$$I_{XX}=\frac{BD^3}{12},$$

then

$$I_{CC}=\frac{BD^3}{12}+BD\left(\frac{D}{2}\right)^2$$

$\left(\text{since } A=BD \quad \text{and} \quad h=\frac{D}{2}\right)$,

$$=\frac{BD^3}{12}+\frac{BD^3}{4}=\frac{BD^3}{3},$$

which we know is correct. (See Table I.)

Example 1.

Find moments of inertia about axes XX and YY (I_{XX} and I_{YY}) of the section shown in Fig. 2. Thickness of plate = 16 S.W.G. = ·064 in.

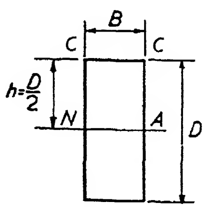


Fig. 1.—To find the moment of inertia about the base CC.

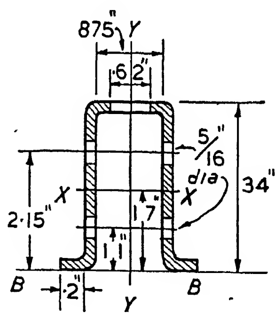


Fig. 2.—To find moments of inertia about XX and YY.

Area.	Moment of Area about BB.
+ ·064 × 255 = + 0163	+ 0163 × 3 368 = + 055
+2 × (·064 × 3 40) = + ·4350	+ 4350 × 1·7 = + ·740
+2 × ·064 × 20 = + 0256	+ 0256 × 032 = + 0008
−4 × ·064 × ·3125 = − ·0800	− ·04 × 2 15 = − 086
= + ·3969 (in.²)	− 04 × 1 1 = − ·044

$\Sigma Ay = +\ 666$ (in ³)

Position of Neutral Axis (XX).

If $\bar{y}$ = distance of XX from BB,

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} - \frac{\cdot 666}{\cdot 3969} = 1{\cdot}7 \text{ in. (say)}$$

Values of I_{XX}.

Moment of Inertia about own Neutral Axis $= \frac{Ad^2}{12}$ (in.^{4}).	Ah^2 (in ^{4})	I_{XX} (in ^{4})
$\frac{0163 \times 064^2}{12}$ = negligible	$0163 \times 1\ 668^2 = 0453$	+ 0453
$\frac{4350 \times 3{\cdot}4^2}{12}$ = 4200	Nil	+ 4200
$\frac{0256 \times 064^2}{12}$ = negligible	$0256 \times 1\ 668^2 = 0712$	+ 0712
$\frac{04 \times {\cdot}3125^2}{12}$ = negligible	$04 \times 45^2 = 0081$	− 0081
$5\ 57 \times 10^{-6}$ = negligible	$04 \times 60^2 = 0144$	− 0144

I_{XX} = 514 (in.^{4}).

To Find I_{YY}.—It is best to divide the figure up into the rectangles

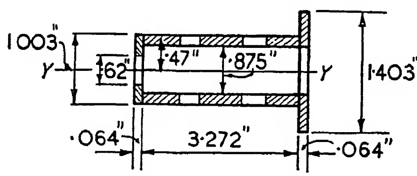


Fig 3 —Division into rectangles

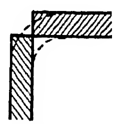


Fig 4 —A close approximation obtained by ignoring the radii and substituting rectangles

shown in Fig. 3. The manner in which a section is divided up, in fact, plays a great part in the ready determination of the moment of inertia.

$$I_{YY} = \frac{3{\cdot}272}{12}[1{\cdot}003^3 - {\cdot}875^3] + \frac{\cdot 064}{12}[1{\cdot}003^3 - {\cdot}62^3] + \frac{\cdot 064}{12}[1{\cdot}403^3 - {\cdot}875^3]$$

$$-4\left[{\cdot}3125 \times \frac{064^3}{12} + {\cdot}3125 \times {\cdot}064 \times {\cdot}47^2\right] = {\cdot}0925 + {\cdot}0154 - {\cdot}0176$$

$$I_{YY} = {\cdot}0903 \text{ (in.}^4\text{)}.$$

It will be noticed in the above example that the corners have been assumed square. An alternative method, which gives a very close approximation to the true state of affairs, is to ignore the radii and substitute the rectangles shown shaded in Fig. 4 to an exaggerated scale.

CHAPTER II.

SHEAR AND BENDING MOMENT DIAGRAMS.

TABLE III shows most of the standard shear and bending moment diagrams for cantilevers and simply supported beams, the notation being· Shear to the *left* of any section is positive when upward; bending moment is positive when putting the top flange in tension.

In practice, the loading does not by any means always give these straightforward cases; it may, for instance, be of the form shown in Example 2, in which one reaction, it will be noted, is in the same direction as the applied loads; or it may consist of a uniformly distributed load and one or more concentrated loads (Example 3), whereupon the individual shear and bending moment (B.M.) diagrams are drawn and added algebraically, i.e. with due regard to sign (positive or negative).

Example 2.

Beam simply supported at A and C, and loaded at B and D as shown (see Fig. 5).

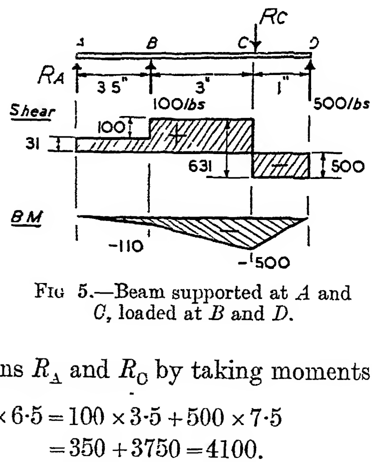


FIG 5.—Beam supported at A and C, loaded at B and D.

First find reactions R_A and R_C by taking moments about A

$$\begin{aligned} R_C \times 6{\cdot}5 &= 100 \times 3{\cdot}5 + 500 \times 7{\cdot}5 \\ &= 350 + 3750 = 4100. \\ R_C &= 631 \text{ lb. (downward).} \\ R_A &= 100 + 500 - 631 = 31 \text{ lb. (upward).} \end{aligned}$$

Care should always be taken with an overhung beam of this type to

make sure that the direction of the reactions is correct The shear diagram in itself, if drawn to scale, serves as a check.

From Fig. 5 we see that the bending moments are·

$$M_B = R_A \times 3{\cdot}5 = -110 \text{ lb. in.}$$
$$M_C = -500 \times 1 = -500 \text{ lb. in.}$$

Check.

$$M_B = -500 \times 4 + 631 \times 3$$
$$= -2000 + 1893$$
$$= -107 \text{ lb. in.}$$

Example 3.

Beam, simply supported at A and B, and with uniformly distributed load and concentrated loads as shown in Fig. 6.

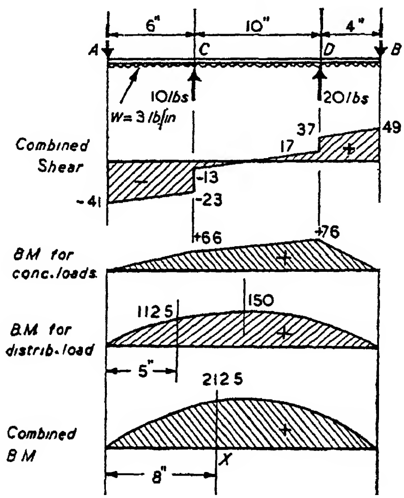


FIG 6.—Beam, supported at A and B, with uniformly distributed load and concentrated loads.

Reactions for combined loading:

$$R_A \times 20 = \frac{3 \times 20^2}{2} + 10 \times 14 + 20 \times 4 = 600 + 140 + 80 = 820.$$
$$R_A \qquad = 41 \text{ lb.}$$
$$R_B \qquad = 60 + 10 + 20 - 41 = 49 \text{ lb.}$$

Reactions for concentrated loads only:

$$20R_{A^1} = 20 \times 4 + 10 \times 14 = 80 + 140 = 220.$$
$$R_{A^1} \quad = 11 \text{ lb.}$$
$$R_{B^1} \quad = 19 \text{ lb.}$$
$$M_{C^1} \quad = R_{A^1} \times 6 = 66 \text{ lb. in.}$$
$$M_{D^1} \quad = R_{B^1} \times 4 = 76 \text{ lb. in.}$$

Table III.—Shear and Bending Moment Diagrams for Cantilevers and Simply Supported Beams.

Cantilevers (cases 1–4)

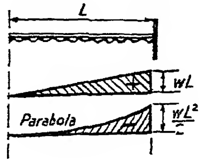
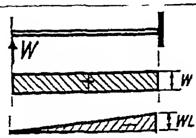
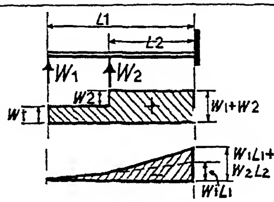
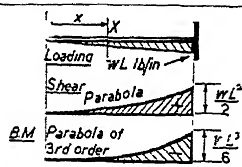
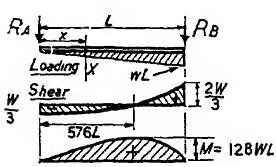
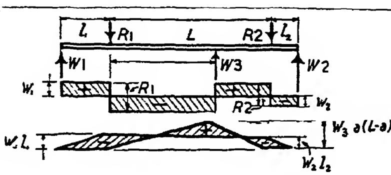
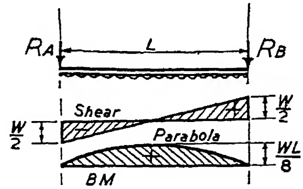
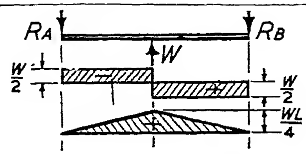
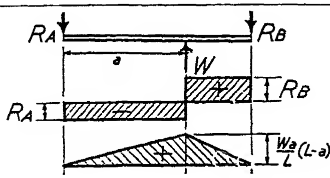
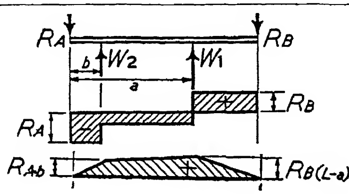
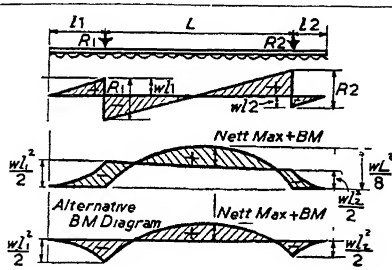
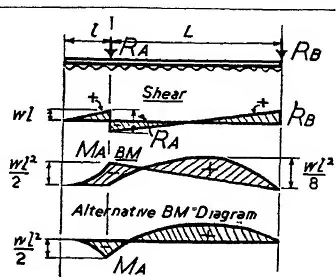
Case.	Shear and B.M. Diagrams.	Remarks
(1) Cantilever Uniformly distributed load w lb./in		Max B.M $=\frac{wL^2}{2}$ at support Deflection $y=\frac{WL^3}{8EI}$ at tip
(2) Cantilever. Concentrated load W.		Max B.M $=WL$ at support $y=\frac{WL^3}{3EI}$ at tip
(3) Cantilever. Two concentrated loads W_1 and W_2		
(4) Cantilever. Uniformly increasing load from 0 at tip to wL lb./in. at root.		$W=\frac{1}{2}wL^2$. Shear at any section $X=\frac{wx^2}{2}$. M at any section $=\frac{wx^2}{2}\times\frac{x}{3}=\frac{wx^3}{6}$.
(9) Simply supported beam. Load uniformly increasing from 0 at one end to wL lb./in. at other.		$W=\frac{1}{2}wL^2$. Shear at section X $=\frac{1}{2}wx^2-R_A$ $=\frac{1}{2}wx^2-\frac{wL^2}{6}$ $=\frac{1}{2}w\left(x^2-\frac{L^2}{3}\right)$ Shear$=0$ when $x^2=\frac{L^2}{3}$ or $x=576L$.
(12) Overhung beam. Concentrated loads W_1, W_2 and W_3		$W=W_1+W_2+W_3$. $R_2=\frac{1}{L}[W_2(L+l_2)+W_3a-W_1l_1]$. $R_1=W-R_2$.

TABLE III—*continued.*

	Case.	Shear and B M Diagrams.	Remarks
Simply Supported Beams	(5) Uniformly distributed load of w lb./in.		$W = wL$ Max. B M. $= \frac{wL^2}{8} = \frac{WL}{8}$ at centre. Deflection $= \frac{5}{384}\frac{WL^3}{EI}$ at centre.
	(6) Concentrated load of W lb at the centre of span.		Max B M. $= \frac{WL}{4}$ at centre Deflection $= \frac{WL^3}{48EI}$ at centre.
	(7) Concentrated load of W lb at distance a from one support.		$R_B = \frac{Wa}{L}$. $R_A = W - R_B$. Max. B M. $= \frac{Wa}{L}(L-a)$ under load.
	(8) Two concentrated loads.		$W = W_1 + W_2$ $R_B = \frac{W_1 a + W_2 b}{L}$. $R_A = W - R_B$
Overhung Beams	(10) Overhung beam, distributed load of w lb./in., unequal overhangs.		$W = w(L + l_1 + l_2)$ $R_2 = \frac{w}{2L}[(L + l_2)^2 - l_1^2]$. $R_1 = W - R_2$. If $l_1 = l_2$, $R_1 = R_2$, and diagrams are symmetrical about the centre line
	(11) Overhung beam with one overhang. Distributed load		$W = w(L + l)$. $R_B = \frac{w}{2L}(L^2 - l^2)$. $R_A = W - R_B$ $M_A = \frac{wl^2}{2}$ $M_B = 0$

For distributed load only:

$$M \text{ at centre} = \frac{wL^2}{8} = \frac{3 \times 20^2}{8} = 150 \text{ lb. in.}$$

$$M \text{ at quarter-span (see below)} = {\cdot}75 \times 150 = 112{\cdot}5 \text{ lb. in.}$$

Check.

Bending moment at any section X between C and D, at say 8 in. from A,

$$M_X = (41 \times 8) - (10 \times 2) - \frac{(3 \times 8^2)}{2}$$

$$= 328 - 20 - 96 = \underline{212 \text{ lb. in.}},$$

which agrees with the value on the combined B.M diagram when drawn to scale.

Bending Moment Diagrams for Uniform Loading.—When drawing the parabolic bending moment diagram for a simply supported beam carrying a uniformly distributed load, it is useful to remember that the B.M. at quarter-span is $\frac{3}{4}\times$ the maximum B.M. at the centre. The proof is:

$$\text{B.M. at quarter-span} = \frac{wL}{2} \times \frac{L}{4} - \frac{w}{2}\left(\frac{L}{4}\right)^2$$

$$= \frac{wL^2}{8} - \frac{wL^2}{32} = \frac{3}{4}\,\frac{wL^2}{8} = \frac{3}{4} M.$$

Similarly for a cantilever with a uniformly distributed load,

B.M. at root $= M$.
B.M. at quarter-span from root $= \frac{9}{16} M$.
B.M. at mid-span $= \frac{1}{4} M$.

If the loading is very irregular and so does not lend itself to the treatment discussed above in Examples 2 and 3, the problem can be solved graphically, but before explaining the method of doing this, the inter-relation of loading, shear, and bending moment will be outlined.

The relation between loading, shear, and bending moment is shown in Fig. 7. Consider a cantilever with a uniformly distributed load of w lb. per inch run. The loading diagram will be a rectangle of ordinate w, and at any section distant x from the tip, the shear, being by definition the sum of the loads to the left, say, of the section, will be wx, the shear at the root being wL. But, as will be seen, the area of the loading diagram up to section X (shown shaded) also equals wx. That is, the shear at any station is represented by the area of the loading diagram up to that station, the total shear

at the root equalling the whole area under the loading curve from tip to root. Similarly, the bending moment at X, $wx \times \frac{x}{2} = \frac{wx^2}{2}$, is the area shaded on the shear curve, so that the B.M. at any section is the area under the shear curve up to there, the value at the root being $\frac{1}{2}wL^2$.

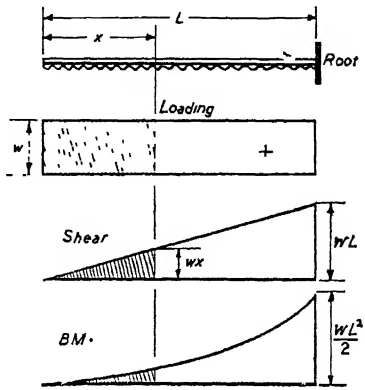


FIG 7.—Diagrams showing the relation between loading, shear, and bending moment

Thus, given the shape of the loading curve on a beam, we can by integration, i.e. by summing the areas under the curve, find the shear, B.M. and, as will be shown, the deflection.

Mathematical Statement.

Proof of the above is given by Fig. 8 in conjunction with the following.

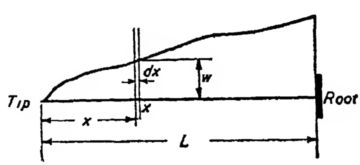


FIG. 8.—Proof for the shear and bending moment calculations.

If w = the loading in lb per inch at any section X of a loading curve, F = shear (lb.) and M = bending moment (lb. in.), then area of element of width $dx = wdx$, and total area of loading curve F up to section $X = \int_0^x wdx$.

Similarly, $$M = \int_0^x Fdx = \iint_0^x wdx \, . \, dx.$$

Example 4.—Graphical Integration for Shear and Bending Moment. (See Fig. 9.)

Consider a typical wing-loading curve such as is given in Air Publication 970, *Design Requirements for Aeroplanes for the Royal Air Force*, Chap. VII, Para. 3, sub-section (iii), Fig. 3, for $\lambda = 1{\cdot}0$.

To construct the shear curve, find the area under the loading curve at various stations by drawing verticals 1–1, 2–2, 3–3, etc (Fig. 9) to divide the loading curve into a series of figures closely approximating to rectangles. When the slope of the curve is changing rapidly, as near the tip in this example, the verticals must be sufficiently close together to make 1–2, 2–3, etc. (on the curve) straight lines, but at stations near the root the spacing can be greater

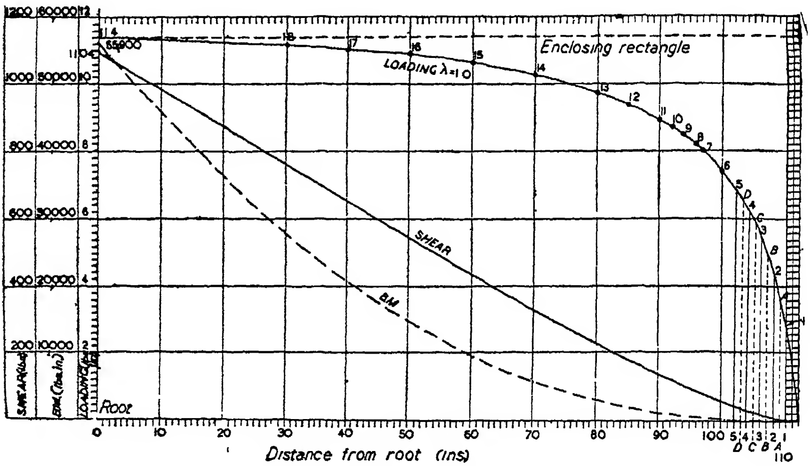


FIG 9.—Graphical integration for shear and bending moment

The area of each rectangle will be the mid-ordinate *AA*, *BB*, etc., multiplied by the base length 1–2, 2–3, etc., respectively.

The results are tabulated in Table IV, the total shear *F* at the root being the sum of the values in column 3, which can be checked against the last value in column 4, since the latter must represent the sum of values in the previous column.

A rough check on the value of *F* is afforded by estimating by eye the area under the loading curve, which in this case is approximately $\frac{7}{8}\times$ the enclosing rectangle $= \frac{7}{8} \times 112 \times 11{\cdot}4 = 1110$ lb.

The shear diagram is the fair curve drawn through the points plotted at various stations from the values in column 4.

Bending Moment Diagram.

By dividing up the shear curve in a similar way and integrating graphically (cols. 5 to 8 inclusive), the bending moment is found and the B.M. diagram plotted.

TABLE IV.—TABULATION FOR SHEAR AND BENDING MOMENT.

Col. 1.	Col 2.	Col 3.	Col. 4.	Col 5.	Col 6.	Col 7	Col. 8
Station (in. from root).	udx.		$\int udx$ up to any Station (lb.).	Station.	Fdx (lb. in).		$\int Fdx$ up to any Station (lb in)
110	$\frac{1}{2}\times 2\times 2{\cdot}4$	2·4	2·4				
108	$3{\cdot}5\times 2$	7·0	9·4				
106	$4{\cdot}95\times 2$	9·9	19·3				
104	$5{\cdot}85\times 2$	11·7	31·0	104	$\frac{1}{2}\times 8\times 30$	120	120
102	$6{\cdot}5\times 2$	13·0	44·0				
100	$7{\cdot}1\times 2$	14·2	58·2	100	45×4	180	300
98	$7{\cdot}6\times 2$	15·2	73·4				
96	$8{\cdot}0\times 2$	16·0	89·4				
94	$8{\cdot}3\times 2$	16·6	106·0				
92	$8{\cdot}6\times 2$	17·2	123·2				
90	$8{\cdot}8\times 2$	17·6	140·8	90	100×10	1,000	1,300
85	$9{\cdot}1\times 5$	45·5	186·3				
80	$9{\cdot}5\times 5$	47·5	233·8	80	185×10	1,850	3,150
70	10×10	100·0	333·8	70	282×10	2,820	5,970
60	$10{\cdot}45\times 10$	104·5	438·3	60	385×10	3,850	9,820
50	$10{\cdot}75\times 10$	107·5	545·8	50	495×10	4,950	14,770
40	$10{\cdot}95\times 10$	109·5	655·3	40	605×10	6,050	20,820
30	$11{\cdot}1\times 10$	111·0	766·3	30	712×10	7,120	27,940
0	$11{\cdot}25\times 30$	337·5	1,103·8	0	932×30	27,960	55,900

$$F=\int_0^L wdx=\underline{1{,}103{\cdot}8\text{ lb.}}\qquad M=\int_0^L Fdx=55{,}900\text{ lb. in.}$$

Slope and Deflection.

Given the bending moment diagram, it is possible to use the method of graphical integration to find the slope and deflection, since we know that

$$\frac{d^2y}{dx^2}=\frac{M}{EI},$$

where

$$E = \text{Young's Modulus}$$

and

$$I = \text{the Moment of Inertia of the section.}$$

$$\text{Slope } \frac{dy}{dx} = \int \frac{M}{EI} dx = \frac{1}{EI} \int M dx, \text{ for constant } E \text{ and } I.$$

$$\text{Deflection } y = \frac{1}{EI} \iint M dx \, . \, dx, \quad \text{or} \quad EIy = \iint M dx \, . \, dx.$$

That is, if the B.M. curve is integrated twice (graphically) we can find EIy and hence y. If E is constant, but I varies at different stations, as is often the case,

$$Ey = \iint \frac{M}{I} dx \, . \, dx.$$

Whereupon we draw the curve of $\frac{M}{I}$ and integrate it twice to find Ey and therefore y. The method will be better understood from the worked example that follows.

Example 5.—Deflection due to Bending of Beam with constant E and I, Loaded as shown in Fig. 10.

The bending moment at various points along the span is calculated—in the general case with irregular loading, the B.M. would be found

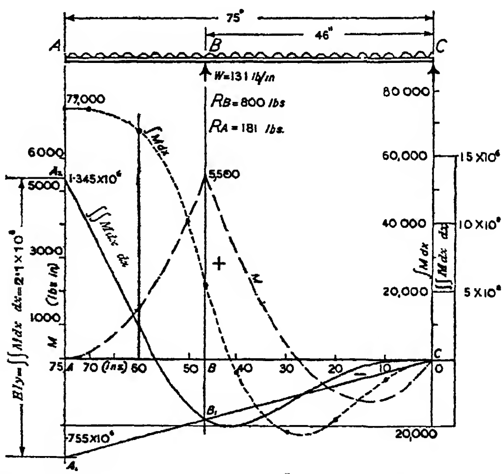


FIG. 10.—Graphs to show deflection due to bending of a beam with constant E and I.

graphically—and the B.M. diagram drawn. By graphical integration, curves of $\int Mdx$ and $\iint Mdx \, . \, dx$ are constructed.

Then, on the assumption that the supports do not move, the deflection at B and C must $=0$, so that if we join C to B_1 (on the curve $\iint Mdx \, . \, dx$) and produce to A_1, the intercept A_1A_2 is a measure of the deflection at A relative to B and C, the value we require. By scaling A_1A_2,

$$EIy = 2{\cdot}1 \times 10^6, \quad \text{or} \quad y = \frac{2{\cdot}1 \times 10^6}{EI}.$$

Taking $E = 1{\cdot}5 \times 10^6$ (spruce)

and $I = 4{\cdot}56$ in.^{4},

deflection $y = {\cdot}307$ in.

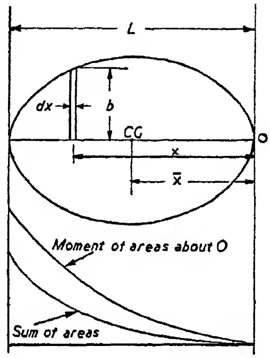


Fig. 11.—Finding the centre of gravity by graphical integration.

Position of the Centre of Gravity of an Area by Graphical Integration (see Fig. 11).—The method of graphical integration discussed above can be applied to find the centre of gravity (C.G.) of any irregular-shaped area. First draw the profile to scale and divide up the figure so as to find the area of each element bdx (see Fig. 11) and plot the "sum of areas" curve so obtained to find the total area of one side $A = \int_0^L bdx$. The complete area will be $2A$.

The integration of this curve gives the moment of the elementary areas about station $O \left(M = \int_0^L x \, . \, dA = \int_0^L xb \, . \, dx \right)$. Then, if $\bar{x}$ is the distance of the C.G. from station O,

$$A\bar{x} = \int_0^L xb \, . \, dx,$$

or

$$\bar{x} = \frac{\int_0^L xb \, . \, dx}{A}.$$

CHAPTER III.

MOMENT OF INERTIA OF THIN ARCS OF CIRCLES.

LET O be the centre of arc distant a from XX (see Fig. 12). The thickness t of the arc is assumed small compared with r. Then

$$I_{XX} = tr\left[2\alpha\left(a^2 + \frac{r^2}{2}\right) + \frac{r^2}{2}\cos 2\theta \sin 2\alpha + 4ar\cos\theta\sin\alpha\right],$$

where I_{XX} = M I about XX and α is in radians.

First moment,

$$A\bar{y} = 2\,tr\,(a\alpha + r\sin\alpha\cos\theta),$$

where

$$A = \text{area} = 2\alpha\, tr;$$

$$\bar{y} = \frac{2\,tr\,(a\alpha + r\sin\alpha\cos\theta)}{2\alpha\, tr}$$

$$= a + \frac{r}{\alpha}\sin\alpha\cos\theta.$$

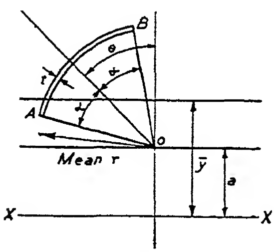


FIG. 12.—To find the moment of inertia of thin arcs of circles.

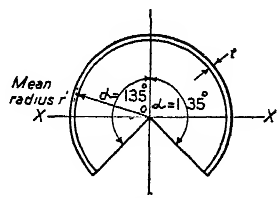


FIG 13 —Diagram to illustrate Example 6.

The worked examples that follow will explain the application of the above formulæ.

Example 6.

Ring of thickness t as shown in Fig. 13

$$\theta = 0.$$

$$\alpha = 135 \text{ degrees} = 2{\cdot}35 \text{ radians}.$$

$$\sin\alpha=\sin(180-135)=\cdot7071.$$
$$\cos\theta=1.$$
$$A\bar{y}=2\,tr\,(a\alpha+r\sin\alpha\cos\theta)$$
$$=2\,tr^2\sin\alpha\cos\theta \qquad (\text{since } a=0)$$
$$=2\,tr^2\times\cdot7071$$
$$=1\cdot4142\,tr^2$$
$$A=2\,t\alpha r$$
$$=2\,tr\times2\cdot35$$
$$=4\cdot7\,tr$$
$$\bar{y}=\frac{1\cdot4142\,tr^2}{4\cdot7\,tr}=\cdot301r.$$

I_{XX}, when $a=0$,

$$=tr^3\left[\alpha+\tfrac{1}{2}\cos2\theta\sin2\alpha\right].$$

But

$$\cos2\theta=\cos0=1$$
$$\sin2\alpha=\sin270=-1$$
$$\therefore\ I_{XX}=tr^3\left[2\cdot35+\tfrac{1}{2}\,.\,1\,.\,-1\right]$$
$$=tr^3\left[2\cdot35-\cdot50\right]$$
$$I_{XX}=1\cdot85\,tr^3.$$

Special Case of the Above : Semicircular Ring.

$$\theta=0 \qquad \cos2\theta=1$$
$$\alpha=90^\circ=\frac{\pi}{2} \qquad \sin2\alpha=0$$

Then $$I_{XX}=tr^3\times\alpha=\frac{\pi\,tr^3}{2}=1\cdot57\,tr^3.$$

Check.

I_{XX} of circular ring from Table I (*ante*)

$$=\frac{\pi}{8}D^3t=\pi\,tr^3,$$

i.e $\quad=\underline{2\times I_{XX}\text{ for semicircular ring.}}$

Example 7.

Find I_{XA} in Fig. 14, where

$$\alpha=45^\circ=\cdot785\text{ radians};$$
$$\sin\alpha=\cdot7071;$$
$$\theta=45.$$

Area:

Length of arc $AB = \cdot 268 \times 2\alpha = \cdot 421$ in.

Area $= \cdot 421 \times \cdot 036 = \cdot 0152$ in.^{2}.

Area of rectangle $BC = \cdot 036 \times \cdot 3078 = \cdot 0111$ in ^{2}.

Total area $= 4 \times 0152 + 8 \times \cdot 0111$.

$A = \cdot 0608 + \cdot 0888 = \underline{\cdot 1496 \text{ in.}^2}$.

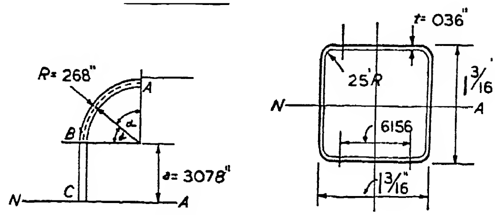


FIG 14.—Finding I about the neutral axis.

Since the section is symmetrical, the position of the neutral axis is known.

For Arc:

$$I_{NA} = tr\left[2\alpha\left(a^2 + \frac{r^2}{2}\right) + \frac{r^2}{2}\cos 2\theta \sin^2 \alpha + 4ar \cos\theta \sin\alpha\right]$$

$$tr = \cdot 036 \times \cdot 268 = \cdot 00965;$$

$$a^2 = \cdot 095;$$

$$\frac{r^2}{2} = \frac{\cdot 072}{2} = \cdot 036,$$

$$\cos 2\theta = 0;$$

$$\sin 2\alpha = 1;$$

$$2\alpha\left(a^2 + \frac{r^2}{2}\right) = 1 \cdot 57(\cdot 095 + \cdot 036) = 1 \cdot 57 \times \cdot 131 = \cdot 206,$$

$$4ar \cos\theta \sin\alpha = 4 \times \cdot 3078 \times \cdot 268 \times \cdot 7071^2 = \cdot 165.$$

$$\therefore I_{NA} = \cdot 00965\, [\cdot 206 + 0 + \cdot 165]$$

$$= \cdot 00965 \times \cdot 371 = \cdot 00358 \text{ in.}^4 \text{ per arc}$$

$$= 4 \times \cdot 00358 = \cdot 01432 \text{ in.}^4 \text{ for 4 arcs.}$$

For Rectangles:

$$2 \times \frac{\cdot 036 \times \cdot 6156^3}{12} = \cdot 006 \times \cdot 234 = \cdot 0014 \text{ in.}^4.$$

$$2 \times \frac{\cdot 6156 \times \cdot 036^3}{12} = \textit{negligible}.$$

$$2 \times (\cdot 6156 \times \cdot 036) \times \cdot 576^2 = \cdot 0147.$$

Total $I_{NA} = \cdot 01432 + \cdot 0014 + \cdot 0147 = \cdot 0304$ in.^{4}.

Example 8.

Find the position of the neutral axis (see Fig. 15).

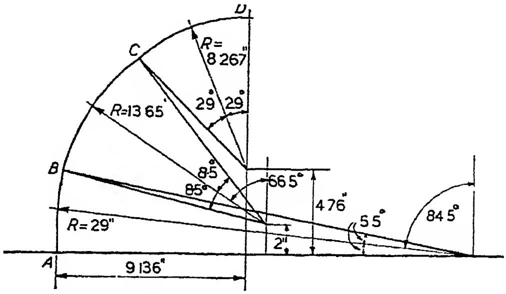


FIG. 15 —Diagram to illustrate Example 8

$$A\bar{y} = 2\,tr\,(a\alpha + r\sin\alpha\cos\theta).$$

	r.	$2tr$.	a.	α.	$a\alpha$	$\sin\alpha$	θ.	$\cos\theta$	$r\sin\alpha\cos\theta$	$A\bar{y}$
			in							
Arc AB	29	2 085	0	09599	0	09585	84 5	09585	2665	·556
„ BC	13 65	·983	2	14835	29670	1478	66 5	39875	·804	1·082
„ CD	8 267	595	4 76	506	2 41	4848	29 0	8746	3 499	3·52

$= 5\ 158$ in.^{3}

$A = 2\alpha\,tr$

Arc AB	·200
„ BC	·146
„ CD	·301

$= ·647$ in.^{2}

Thus $\bar{y} = \dfrac{5·158}{·647} = 7\ 98$ in.

TEST EXAMPLES ON PRECEDING MATTER.

The following are further examples on the work which has been covered in preceding pages. Answers are given separately at the end of the chapter.

(1) Find the moment of inertia of the built-up section shown in Fig. 16. All dimensions are in inches.

(2) The overhung beam shown in Fig. 17 has the following values of I at various stations :—

Dist. from C (in.).	0	10	15	20	30	40	46	50	60	65	70	75
I (in.^{4}).	5 94	6 6	6 9	7·25	7 85	8 45	8 56	8 40	7 45	6·95	6 45	5 94

If $E = 1{\cdot}5 \times 10^6$, find the deflection at A.

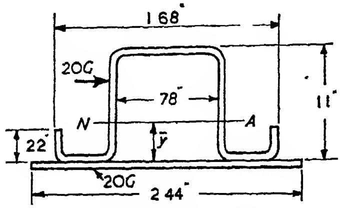


FIG. 16.—Built-up section.

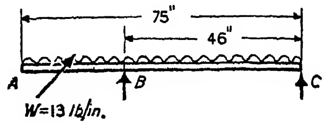


FIG. 17—Overhung beam, for which it is necessary to find the deflection at A.

(3) Find I about the neutral axis of the section shown in Fig. 18.

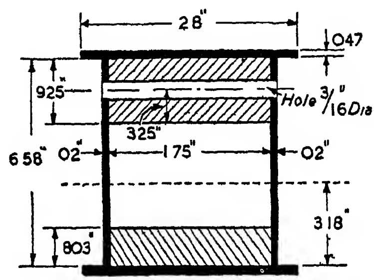


FIG. 18.—Section for Example (3).

(4) The semi-ordinates of a longitudinal cross-section at various distances from the bow of a float are given in the accompanying table. Determine the position of the centre of gravity by graphical integration.

Ins. from Bow.	$11\frac{3}{4}$	$18\frac{1}{2}$	$24\frac{1}{8}$	$27\frac{5}{8}$	$37\frac{1}{8}$	$49\frac{3}{8}$	$62\frac{3}{8}$	$75\frac{1}{4}$	$93\frac{1}{8}$	$111\frac{5}{8}$	$124\frac{1}{8}$
Semi-Ordinate (in.).	0	$3\frac{5}{8}$	$8\frac{1}{4}$	$12\frac{1}{2}$	$12\frac{1}{2}$	$12\frac{1}{2}$	$12\frac{1}{4}$	$12\frac{1}{8}$	$11\frac{3}{4}$	$11\frac{1}{2}$	$11\frac{1}{4}$

Ins from Bow.	$137\frac{1}{8}$	$147\frac{7}{8}$	$166\frac{3}{8}$	$183\frac{1}{2}$	$189\frac{1}{2}$	$195\frac{1}{2}$	$197\frac{1}{2}$	$199\frac{1}{2}$	$201\frac{1}{2}$	$203\frac{1}{2}$	Stern
Semi-Ordinate (in.).	$10\frac{5}{8}$	$10\frac{1}{8}$	$8\frac{3}{4}$	$7\frac{1}{4}$	$6\frac{1}{2}$	$5\frac{1}{2}$	5	$4\frac{3}{8}$	$3\frac{5}{8}$	$2\frac{1}{4}$	0

Answers :

(1) $\bar{y} = {\cdot}31$ in.; $I_{NA} = {\cdot}0398$ in.^{4}.
(2) $Ey = 232{,}900$; $y = {\cdot}155$ in.
(3) $I = 26{\cdot}33$ in.^{4}; N.A. $= 3{\cdot}18$ in.
(4) $\bar{x} = 104$ in. from stern; $A = 3838$ in.^{2}.

CHAPTER IV.

FIXED AND CONTINUOUS BEAMS.

THE essential difference between a fixed beam and one which is freely supported is that the former has one or both ends constrained so that the normal deflection of the beam under load is reduced.

In standard text-books on structural engineering it is usual to consider that the end-fixing is due to building the beam into a wall, but in aircraft structures it is better to think of the constraint as a fixing moment applied at the support by the attachment fitting, whatever it may be.

Standard cases of Fixed Beams are given in Table V.

Continuous Beams—Clapeyron's Theorem of Three Moments.

(Neglecting end load and assuming constant I and that the supports do not sag.)

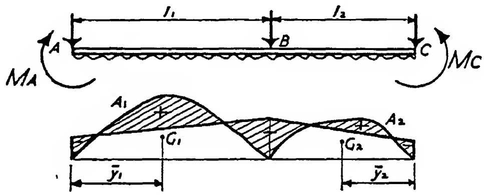


FIG. 19.—Diagrams to illustrate Clapeyron's Theorem.

If G_1 and G_2 (see Fig. 19) are the centroids of the Free Bending Moment curves,

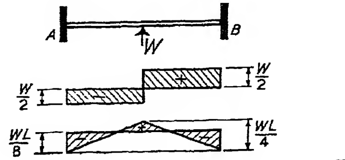
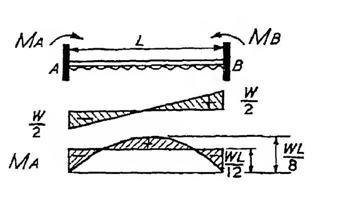
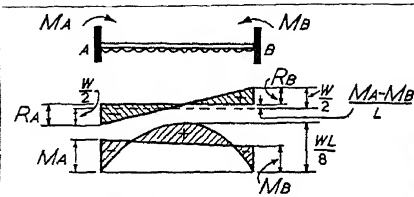
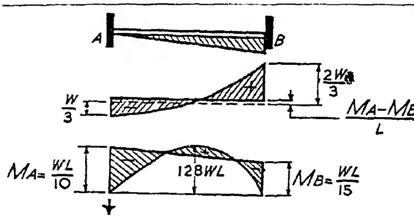
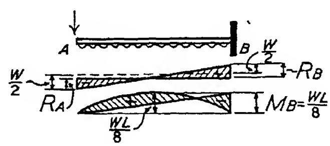
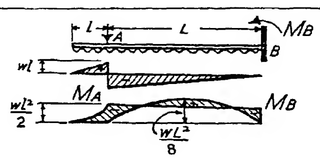
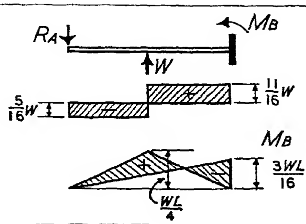
$\bar{y}_1$ and $\bar{y}_2$ are the distances of their C.G.'s from A and C respectively, and

A_1 and A_2 are the areas of the Free B.M. curves,

then

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = 6\left(\frac{A_1 \bar{y}_1}{l_1} + \frac{A_2 \bar{y}_2}{l_2}\right) \qquad . \quad . \quad . \quad (1)$$

$$= \tfrac{1}{4}(w_1 l_1^3 + w_2 l_2^3). \qquad . \quad . \quad . \quad (2)$$

Case	Shear and B M. Diagrams	Remarks
(13) Concentrated load of W lb. at centre.		Shear diagram is as for simply supported beam. B M is maximum at centre or at supports $=\dfrac{WL}{8}$. $y=\dfrac{WL^3}{192EI}$ at centre
(14) Uniformly distributed load of w lb./in. ($M_A=M_B$)		$W=wL$. Since $M_A=M_B$. Shear diagram is as for simply supported beam (*cf.* Case 5). Net B M at centre $=\dfrac{WL}{24}$. Max B M $=\dfrac{WL}{12}$ at supports. $y=\dfrac{WL^3}{384EI}$ at centre
(15) Uniformly distributed load of w lb./in. ($M_A > M_B$)		$W=wL$. $R_A=\dfrac{W}{2}+\left(\dfrac{M_A-M_B}{L}\right)$. $R_B=\dfrac{W}{2}-\left(\dfrac{M_A-M_B}{L}\right)$. Maximum B.M. will depend on relative values of w, M_A and M_B
(16) Load uniformly increasing from 0 at one end to wL at the other.		$R_A=\dfrac{W}{3}+\dfrac{M_A-M_B}{L}$. $R_B=\dfrac{2W}{3}-\dfrac{M_A-M_B}{L}$. (*Cf* Case 9.)
(17) Cantilever propped at end. Distributed load of w lb./in. (*Cf* Case 15 with $M_A=0$.)		$W=wL$. $R_A=\dfrac{W}{2}-\dfrac{M_B}{L}=\dfrac{3}{8}W$. $R_B=\dfrac{W}{2}+\dfrac{M_B}{L}=\dfrac{5}{8}W$. $M_A=0$. $M_B=\dfrac{WL}{8}$
(18) Cantilever propped at intermediate point. (*Cf.* Case 11.)		$W=w(L+l)$. $R_A=\dfrac{w}{2L}(L+l)^2-\dfrac{M_B}{L}$. $R_B=W-R_A$. $M_A=\dfrac{wl^2}{2}$
(19) Cantilever propped at end. Isolated load W lb. at the centre.		$R_A=\dfrac{W}{2}+\dfrac{M_A-M_B}{L}=\dfrac{5}{16}W$. $R_B=\dfrac{11}{16}W$. $M_A=0$. $M_B=\dfrac{3WL}{16}$.

If A and C are simply supported, i.e. if there is no constraint there, $M_A = M_C = 0$, thus simplifying the above equation considerably.

Example 9.—Continuous Beam, Loaded as in Fig. 20.

The fixing moments at A and C can be found directly by considering the cantilever portions DA and EC respectively.

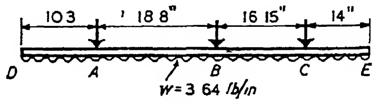


Fig. 20.—Continuous beam, supported at A, B and C, and carrying a uniformly distributed load of 3 64 lb /in.

$$M_A = \frac{3{\cdot}64 \times 10{\cdot}3^2}{2} = 193 \text{ lb. in.}$$

$$M_C = \frac{3{\cdot}64 \times 14^2}{2} = 357 \text{ lb. in.}$$

$$M_A l_1 + 2M_B (l_1 + l_2) + M_C l_2 = \frac{w}{4}(l_1{}^3 + l_2{}^3).$$

Since $w_1 = w_2$, and

$$\begin{aligned} l_1 &= 18{\cdot}8 & l_1{}^3 &= 6{,}650 \\ l_2 &= 16{\cdot}15 & l_2{}^3 &= 4{,}230 \\ l_1 + l_2 &= \underline{\overline{34{\cdot}95}} & l_1{}^3 + l_2{}^3 &= \underline{\overline{10{,}880}} \end{aligned}$$

$$\therefore\ 193 \times 18{\cdot}8 + 2M_B \times 34{\cdot}95 + 357 \times 16{\cdot}15 = \frac{3{\cdot}64}{4} \times 10{,}880$$

$$3{,}620 + 69{\cdot}9M_B + 5{,}760 = 9{,}890$$

$$M_B = \frac{510}{69{\cdot}9} = 7 \text{ lb. in.}$$

Free bending moment at mid-point of AB

$$= \frac{3{\cdot}64 \times 18{\cdot}8^2}{8} = 161 \text{ lb. in.}$$

$$\text{and at } \tfrac{1}{4}AB = 121 \text{ lb. in.}$$

Free bending moment at mid-point of BC

$$\frac{3{\cdot}64 \times 16{\cdot}15^2}{8} = 119 \text{ lb. in.}$$

$$\text{and at } \tfrac{1}{4}BC = 89 \text{ lb. in.}$$

Reactions

Take moments about B of forces to the left of that point in Fig. 20.

$$R_A \times 18{\cdot}8 - \frac{3{\cdot}64}{2} \times (18{\cdot}8 + 10{\cdot}3)^2 + M_B = 0.$$

$$18\ 8R_A = \frac{3\ 64}{2} \times 29{\cdot}1^2 - 7$$

$$= 1540 - 7 = 1533$$

$$R_A = 81{\cdot}5 \text{ lb.}$$

Similarly,

$$R_C = \frac{1}{16{\cdot}15}\left(\frac{3{\cdot}64}{2} \times 30{\cdot}15^2 - 7\right) = 102 \text{ lb.}$$

$$R_B = 3\ 64 \times 59{\cdot}25 - (81{\cdot}5 + 102)$$

$$= 216 - 183{\cdot}5 = \underline{32{\cdot}5 \text{ lb.}}$$

To Draw the Shear Diagram.

At A (Fig. 21), shear from cantilever DA

$$= 3{\cdot}64 \times 10{\cdot}3$$
$$= +37{\cdot}5 \text{ lb.}$$
$$R_A = -81{\cdot}5 \text{ lb.}$$
$$\text{Difference} = -44 \text{ lb.}$$

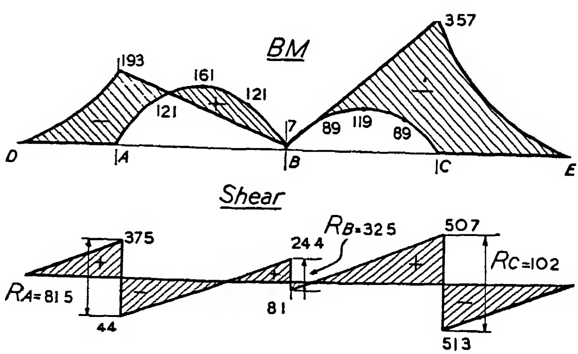


FIG. 21.—Bending moment and shear diagrams.

Similarly at C, shear from cantilever

$$= -3{\cdot}64 \times 14$$
$$= -51{\cdot}3 \text{ lb.}$$
$$R_C = +102 \text{ lb.}$$
$$\text{Difference} = +50{\cdot}7 \text{ lb.}$$

Since the loading is constant, the slope of the shear curve will be the same throughout. In length $AB=18{\cdot}8$ in., shear will alter $3{\cdot}64 \times 18{\cdot}8 = 68{\cdot}4$ lb.

I.e. intercept at $B = 68{\cdot}4 - 44 = 24{\cdot}4$.

$$R_B = 32{\cdot}5 \text{ lb.}$$
$$\text{Difference} = 8{\cdot}1 \text{ lb.}$$

The shear curve can now be drawn.

As a check on the shear curve, the total areas above and below the base line must be equal.

Special Case of Continuous Beam over Two Equal Spans.

If $l_1 = l_2$ and w = constant in Fig. 22, $M_A = M_C = 0$ (from Clapeyron).

$$\therefore\ 4M_B l = \frac{w}{2} l^3$$

$$M_B = \frac{wl^2}{}$$

Reactions

$$R_A l - \frac{wl^2}{2} + M_B = 0$$
$$R_A = \tfrac{3}{8} wl$$
$$R_C = \tfrac{3}{8} wl$$
$$R_B = 2wl - \tfrac{3}{4} wl = \tfrac{5}{4} wl.$$

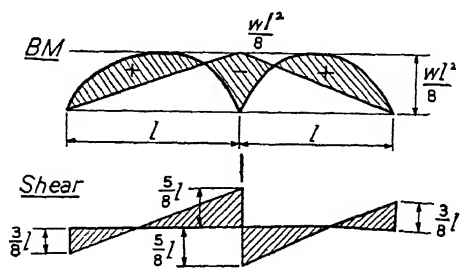


FIG. 22.—Diagrams for a continuous beam over two equal spans.

Continuous Beam—Isolated Load on One Span (see Fig 23).

Free bending moment on BC:

$$R_C = \frac{462 \times 4}{17{\cdot}75} = 104 \text{ lb.}$$

$$R_B = 358 \text{ lb.}$$

Free bending moment at $D = 104 \times 13{\cdot}75 = 1432$ lb. in.

The free B.M. is now drawn and $\bar{y}_2$ measured.

Then from Clapeyron's Theorem, since M_A, M_C and $A_1 = 0$,

$$2M_B(l_1 + l_2) = 6\frac{A_2\bar{y}_2}{l_2}$$

$$\frac{6\left(\frac{1}{2} \cdot 1432\, l_2\right)10{\cdot}5}{l_2} = 45{,}100$$

$$M_B = \frac{45{,}100}{71} = 636 \text{ lb. in.}$$

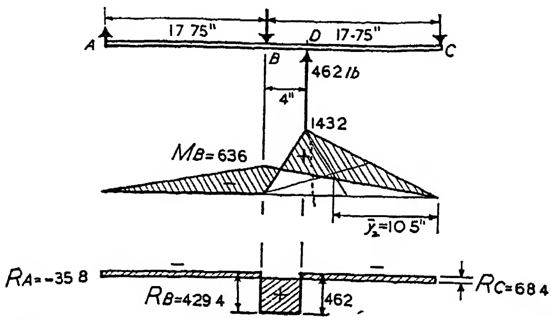


Fig. 23.—Diagrams for a continuous beam with an isolated load on one span

Reactions.

Moments about B of forces to the right

$$R_C \times 17{\cdot}75 - 462 \times 4 + 636 = 0.$$

$$R_C = \frac{1212}{17{\cdot}75} = 68{\cdot}4 \text{ lb.}$$

Moments about B of forces to the left:

$$R_A = \frac{-636}{17{\cdot}75} = -35{\cdot}8 \text{ lb.}$$

$$R_B = 462 - 68{\cdot}4 + 35{\cdot}8$$
$$= 429{\cdot}4 \text{ lb.}$$

Generalized Equation of Three Moments without End Load.

Air Publication 970, VI, 6, (i) gives:

$$\frac{a_1}{I_1}M_A f(a_1) + \frac{a_2}{I_2}M_C f(a_2) + 2M_B\left\{\frac{a_1}{I_1}\phi(a_1) + \frac{a_2}{I_2}\phi(a_2)\right\} = \frac{w_1 a_1^3}{I_1}\phi(a_1) + \frac{w_2 a_2^3}{I_2}\phi(a_2)$$

From the Berry Functions it may be seen that when $P_1 = P_2 = 0$,

$$\alpha_1 = a_1 u_1 = a_1 \sqrt{\frac{P_1}{EI_1}} = 0, \quad \text{and} \quad f(\alpha_1) = 1\cdot 0$$

$$\alpha_2 = a_2 u_2 = a_2 \sqrt{\frac{P_2}{EI_2}} = 0, \quad \text{and} \quad f(\alpha_2) = 1\cdot 0$$

(This is because $\frac{0-1}{0}$ is infinite.)

The expression thus simplifies to—

$$\frac{a_1}{I_1} M_A + \frac{a_2}{I_2} M_C + 2M_B\left(\frac{a_1}{I_1} + \frac{a_2}{I_2}\right) = \frac{w_1 a_1^3}{I_1} + \frac{w_2 a_2^3}{I_2}.$$

Substituting $\frac{l_1}{2}$ for a_1 and $\frac{l_2}{2}$ for a_2 (a_1 and a_2 are the semi-spans), we get—

$$M_A \frac{l_1}{I_1} + M_C \frac{l_2}{I_2} + 2M_B\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) = \tfrac{1}{4}\left(\frac{w_1 l_1^3}{I_1} + \frac{w_2 l_2^3}{I}\right)$$

which reduces to the standard form of Clapeyron when $I_1 = I_2$.

The following worked example will show the application of the method, as well as indicating the treatment for a continuous beam of more than two spans.

Example 10.—Continuous Beam with constant w but varying I. (See Fig. 24.)

$$w_1 = w_2 = w = 2\cdot 72 \text{ lb./in.}$$

$$M_D = 2\cdot 72 \times 14\cdot 5^2 = 286 \text{ lb. in.}$$

$$M_A = 0.$$

Consider spans AB and BC.

$$0 + M_C \times \frac{26}{2\cdot 54} + 2M_B\left(\frac{26\cdot 5}{3\cdot 09} + \frac{26}{2\cdot 54}\right) = \frac{2\cdot 72}{4}\left(\frac{26\cdot 5^3}{3\cdot 09} + \frac{26^3}{2\cdot 54}\right)$$

$$10\cdot 22 M_C + 37\cdot 6 M_B = \cdot 68\,(6020 + 6940)$$

$$= 8810$$

$$M_C + 3\cdot 68 M_B = 861 \qquad (1)$$

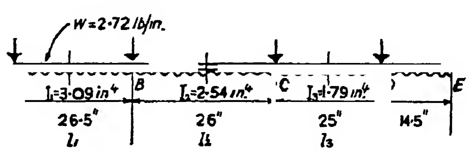


FIG. 24.—Continuous beam with constant w but varying I.

Consider spans BC and CD.

$$+M_D\frac{l_3}{I}+2M_C\left(\frac{l_2}{I_2}+\frac{l_3}{I_3}\right)=\cdot 68\left(\frac{l_2^3}{I_2}+\frac{l_3^3}{I}\right.$$

$$10\cdot 22M_B+286\times\frac{25}{1\cdot 79}+2M_C\left(10\cdot 22+\frac{25}{1\cdot 79}\right)=\cdot 68\,(6940+8750)$$

$$10\cdot 22M_B+4000+48\cdot 44M_C=10{,}680$$

$$10\cdot 22M_B+48\cdot 44M_C=6680 \quad . \quad . \quad . \quad (2$$

whence $M_B=\underline{208\text{ lb in.}}$

and $M_C=\underline{94\text{ lb. in.}}$

Free bending moments ·

$$\text{On } AB=\frac{2\cdot 72}{8}\times 26\cdot 5^2=238\text{ lb in}$$

$$\text{On } BC=\frac{2\cdot 72}{8}\times 26^2\quad=230\text{ lb. in.}$$

$$\text{On } CD=\frac{2\cdot 72}{8}\times 25^2\quad=213\text{ lb. in.}$$

Reactions.

Moments to the left of B.

$$R_A\times 26\cdot 5=\frac{2\cdot 72\times 26\cdot 5^2}{2}-208$$

$$=952-208=744$$

$$R_A=28\text{ lb.}$$

Moments to the left of C:

$$R_B\times 26=\frac{2\cdot 72\times 52\cdot 5^2}{2}-\overset{R_A}{28}\times 52\cdot 5-\overset{M_C}{94}$$

$$=3760-1470-94=2196$$

$$R_B=\underline{84\text{ lb.}}$$

Moments to the right of C.

$$R_D\times 25=\frac{2\cdot 72\times 39\cdot 5^2}{2}-\overset{M_C}{94}$$

$$=2120-94=2026$$

$$R_D=\underline{81\text{ lb.}}$$

$$R_C=2\cdot 72\times 92\cdot 0-(28+84+81)$$

$$=250-193=\underline{57\text{ lb.}}$$

Example 11.—Continuous Beam over Four Spans with a Uniformly Increasing Load.

As an approximation, we can assume that the B.M. at C (see Figs. 25 and 26) is the same as at the supports of a fixed beam; that is, $M_C = \frac{wl^2}{12}$.

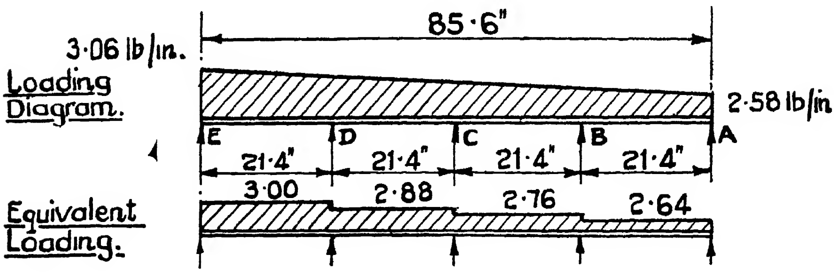


FIG. 25.—Loading diagrams.

Taking

$$w = \frac{2{\cdot}88 + 2{\cdot}76}{2} = 2{\cdot}82 \text{ lb./in.},$$

$$M_C = 2{\cdot}82 \times \frac{21{\cdot}4^2}{12} = \underline{108 \text{ lb. in.}}$$

and

$$M_A = M_E = 0.$$

For spans AB and BC:

$$2M_B \times 2 \times 21{\cdot}4 + 108 \times 21{\cdot}4 = \frac{21{\cdot}4^3}{4}(2{\cdot}64 + 2{\cdot}76)$$

$$4M_B + 108 = 460 \times \frac{5{\cdot}4}{4} = 620$$

$$M_B = \underline{128 \text{ lb. in.}}$$

For spans CD and DE:

$$2M_D \times 2 \times 21{\cdot}4 + 108 \times 21{\cdot}4 = \frac{21{\cdot}4^3}{4}(3{\cdot}0 + 2{\cdot}88)$$

$$M_D = \underline{142{\cdot}5 \text{ lb. in.}}$$

Reactions.

By moments about D—

$$R_E \times 21{\cdot}4 - 3{\cdot}0 \times \frac{21{\cdot}4^2}{2} + 143 = 0$$

$$R_E = \underline{25 \text{ lb.}}$$

By moments about C—

$$R_E \times 2 \times 21{\cdot}4 + R_D \times 21{\cdot}4 - 3\ 00 \times 21{\cdot}4 \times 32{\cdot}1 - 2{\cdot}88 \times \frac{21{\cdot}4^2}{2} + 108 = 0$$

$$R_D = \underline{72\ \text{lb.}}$$

By moments about B—

$$R_A \times 21{\cdot}4 - 2{\cdot}64 \times \frac{21\ 4^2}{2} + 128 = 0$$

$$R_A = 22\ \text{lb.}$$

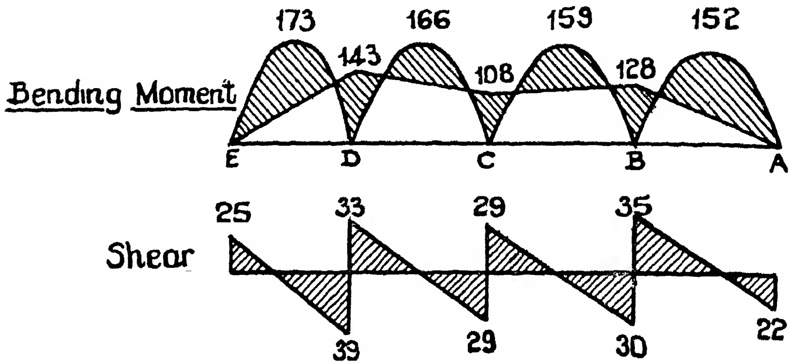


Fig. 26.—Bending moment and shear diagrams

By moments about C—

$$R_A \times 2 \times 21{\cdot}4 + R_B \times 21{\cdot}4 - 2{\cdot}64 \times 21{\cdot}4 \times 32{\cdot}1 - 2{\cdot}76 \times \frac{21{\cdot}4^2}{2} + 108 = 0$$

$$R_B = \underline{65\ \text{lb.}}$$

Total load $= 21{\cdot}4\ (3{\cdot}0 + 2{\cdot}88 + 2{\cdot}76 + 2{\cdot}64) = 242$ lb.

$$\therefore\ R_C = 242 - (22 + 65 + 72 + 25) = 242 - 184 = \underline{58\ \text{lb.}}$$

Free bending moments—

$$\frac{wl^2}{8} = 57{\cdot}5w$$

on $DE = 173$ lb. in.
$CD = 166$,,
$BC = 159$,,
$AB = 152$,,

Flight (*The Aircraft Engineer*), July 3, 1941. "Continuous Beams," by the author.

CHAPTER V.

STRUTS.

CONSIDER a column or strut of length l with an axial compressive end load P, of sufficient magnitude to make the strut deflect an amount y at a distance x from one end.

We require to find what value of P is permissible before causing the strut to fail in bending due to its deflection, on the assumption that:

(a) the ends are pin-jointed;
(b) the load is initially applied along the neutral axis of the strut;
(c) the N.A. is perfectly straight when the strut is unloaded;
(d) the strut is of constant section throughout its length.

The extent to which these assumptions are correct will be discussed later.

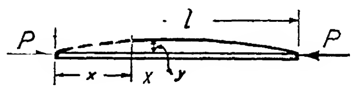


FIG. 27.—A strut axially loaded and pin-jointed at each end.

Euler's Crippling Load for a Strut axially loaded and pin-jointed at each end.

Referring to Fig. 27, we see that

l = length between pin centres;
P = compressive end load; and
y = deflection at distance x from one end.

Bending moment at X is:

$$EI\frac{d^2y}{dx^2} = -Py.$$

Now
$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y = -\mu^2 y,$$

where

$$\mu^2 = \frac{P}{EI} \quad \text{or} \quad \mu = \sqrt{\frac{P}{EI}}$$

The solution of this differential equation is

$$y = A \sin \mu x + B \cos \mu x,$$

and when $x = 0$, $y = 0$. $\therefore$ $B = 0$, i.e.

$$y = A \sin \mu x.$$

When $x = l$, $y = 0$. $\therefore$ $0 = A \sin \mu l$, or $\sin \mu l = 0$, so that

$$\mu l = 0, \quad \pi, \quad 2\pi, \quad \text{etc.}$$

Consider the solution $\mu l = \pi$:

$$\mu^2 = \frac{\pi^2}{l^2}$$

and

$$\frac{P}{EI} = \frac{\pi^2}{l^2}, \quad \text{or} \quad P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EAk^2}{l^2},$$

since

$$I = Ak^2.$$

Allowable stress:

$$p = \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}.$$

Example 12.—Euler Strut Curve for Tube of Various Lengths.

Consider duralumin tube, Specification T.4 ($E = 10 \cdot 5 \times 10^6$), $1\frac{1}{4}$ in. outside diameter (O/D) ×17G., of various lengths l;

$$A = \cdot 2100$$
$$k = \cdot 4226$$
$$\pi^2 E = 103\ 4 \times 10^6.$$

On plotting the values of allowable crippling stress (p) against l/k (see Table VI and Fig. 28), it is seen that, as l/k reaches about 50, p increases rapidly (in the limit when $l = 0$, p would be infinity); but the value of p clearly cannot exceed the allowable stress for the material, represented by BE_2. It follows then that, for small values of l/k, the curve $DCBE_2$ would be more indicative of the state of affairs, and that Euler's formula does not hold for short struts. It is, in fact, not used in practice for values of l/k less than 130. Many other formulæ, notably Southwell's, are used instead.

This formula, in the form given in Air Publication 970, VIII, I, 1, is.

$$p_2 = \frac{P}{A} + \frac{Peh \sec a}{Ak^2}$$

for values of

$$\frac{d}{t} < 80,$$

TABLE VI.—VALUES FOR EULER STRUT CURVES.

l (in.).	$\frac{l}{k}$	$\left(\frac{l}{k}\right)^2$	Compressive Stress $p=\frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$ (lb./in.²)
10	24	570	185,000
13	31	960	107,800
15	36	1,300	79,500
20	47	2,250	46,000
22·5	53	2,810	36,800
25	59	3,480	29,700
30	71	5,050	20,500
40	95	9,000	11,500
50	118	14,000	7,390
60	142	20,000	5,170

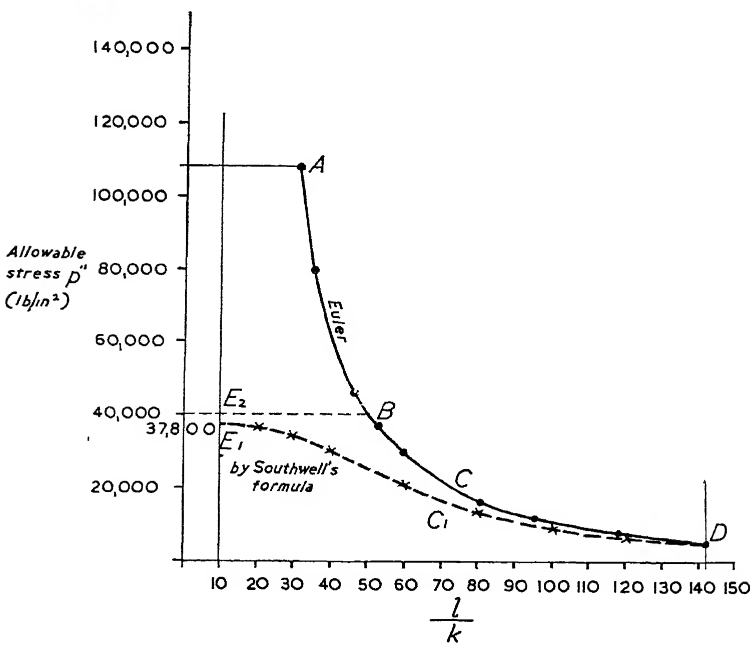


FIG. 28.—Allowable crippling stress against l/k curves according to Euler's and Southwell's formulæ.

where

$p_2 = 0{\cdot}2$ per cent. proof stress,

$P =$ crippling load of strut,

$e =$ equivalent eccentricity of end load,

$h =$ distance from the normal position of N.A. to the most highly stressed fibre, and

$$a = \frac{l}{2}\sqrt{\frac{P}{EI}}.$$

Actual Strut Curves.—As regards the assumptions made earlier in discussing Euler's formula:

The ends are frequently not pin-jointed, there being a fixing moment applied by the end fitting. This is taken account of by assuming an equivalent length of strut less than the length between the attachments, the value of the assumed length depending on the nature of the design in any particular case. For example, for fixity at one end use $0{\cdot}9\,l$, for fixity at both ends use $0{\cdot}8\,l$.

The load is offset from the centre, either due to initial eccentricity of manufacture or by reasons of design, or both. The term e in Southwell's formula takes account of this.

Southwell's formula may be rewritten in the form·

$$\text{allowable stress } p = \frac{p_2}{1 + \lambda \sec \dfrac{l}{2k}\sqrt{\dfrac{p}{E}}},$$

where

$$= \frac{}{k^2},$$

which it is laborious to solve, since p appears on each side of the equation.

However, l/k curves based on this or some other formula are available in a stress office (strut curves for steel (T.45) and duralumin (T.4) tubes are given in Fig. 29), but failing these, the method given in Air Publication 970, VIII, I, 3, Fig. 2, and illustrated therein by a worked example, can be used.

Thus, in any given case of a strut with compressive end load only (note this proviso), find the ratio l/k from the dimensions of the strut on the drawing, bearing in mind the end-fixing conditions, and read off from the curve the allowable stress, compare with the actual factored stress (end load/area) on the member, which must, of course, be less. This is done in the worked example that follows (Example 13).

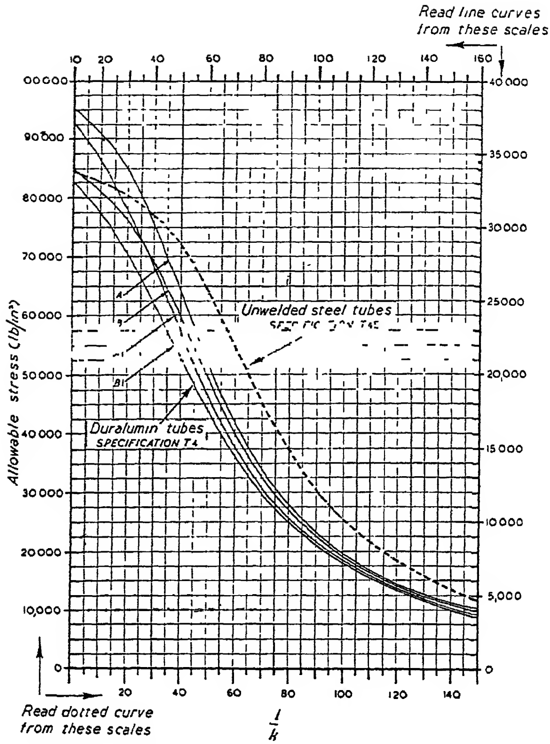


FIG. 29.—Allowable stress against l/k curves for steel T.45 and duralumin T.4 tubes.

A $\frac{3}{4}$-in O/D and over, as received.
B ,, ,, , heat treated.
A_1 less than $\frac{3}{4}$-in O/D, as received.
B_1 ,, ,, , heat treated

Example 13.

A member 23 in. long has to be designed to carry 2443 lb. (tension) or 756 lb. (compression). It is proposed to use a $\frac{3}{4}$-in. O/D × 20G T.45 tube. Will this be up to strength?

Take allowable tensile stress = 101,000 lb /in.²

$$\frac{l}{k} = \frac{23}{\cdot 2528} = 91.$$

Allowable compressive stress (from dotted strut curve) = 30,000 lb./in.²

$$\text{Actual compressive stress} = \frac{756}{\cdot 0807} = 9380 \text{ lb./in.}^2$$

$$\text{Reserve Factor (R.F.)} = \frac{30{,}000}{9380} = 3{\cdot}2$$

$$\text{Actual tensile stress } \frac{2443}{{\cdot}0807} = 30{,}200 \text{ lb /in.}^2.$$

$$\text{R F.} = \frac{101{,}000}{30{,}200} = \underline{3{\cdot}35}.$$

The tube is therefore satisfactory.

In practice the rivet, not the tube strength, would often be the criterion for design.

When the loading on the strut consists of a lateral load and/or a bending moment, as well as a compressive end load, a Howard diagram must be drawn, as will be explained below

Strut with End Load and Lateral Load.—Consider a strut of length $2a$ with end load P and distributed lateral load w lb./in. (see Fig. 30). It is merely a matter of convenience to call the length $2a$ and not l, this will be clear as we proceed For a simply supported beam with lateral load only, the B.M. diagram would be of the usual parabolic form, having a maximum ordinate $\frac{w}{8}(2a)^2$ at the centre O.

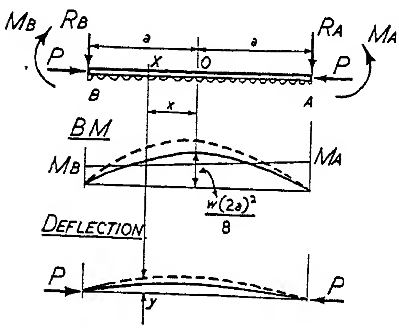


FIG. 30.—Diagrams for a strut of length $2a$ with end load P and distributed lateral load of w lb./in.

The effect of the end load is to increase the deflection, and bending moment, at all points, as shown dotted in Fig. 30.

If M = the *true* bending moment at any section $= EI\dfrac{d^2y}{dx^2}$,

and S = the *true* shear at any section $= \dfrac{dM}{dx}$,

then at section X, distant x from the centre of the strut as shown, the total deflection is y, and the true bending moment is—

$$M = \frac{w(a-x)^2}{2} + M_B - R_B(a-x) - Py.$$

Differentiating once, we get—

$$\frac{dM}{dx} + \frac{P \, . \, dy}{dx} = \frac{w}{2}(-2a+2x) + R_B;$$

and differentiating again—

$$\frac{d^2M}{dx^2}+\frac{P.d^2y}{dx^2}=w.$$

But

$$\frac{d^2y}{dx^2}=\frac{M}{EI};$$

$$\therefore \frac{d^2M}{dx^2}+\frac{PM}{EI}=w.$$

Putting $\mu^2=\frac{P}{EI}$,

$$\frac{d^2M}{dx^2}+\mu^2M=w \quad . \quad . \quad (1)$$

The solution of this equation is—

$$M=A\sin\mu x+B\cos\mu x+\frac{w}{\mu^2} \quad . \quad . \quad (2)$$

or

$$M-\frac{w}{\mu^2}=A\sin\mu x+B\cos\mu x \quad . \quad (2a)$$

Putting $m=M-\frac{w}{\mu^2}$,

$$m=A\sin\mu x+B\cos\mu x,$$

$$m=C\cos(\mu x-\epsilon) \quad . \quad . \quad . \quad (3)$$

or

$$M=C\cos(\mu x-\epsilon)+\frac{w}{\mu^2} \quad . \quad . \quad (3a)$$

Shear.

$$S=\frac{dM}{dx}=-\mu C\sin(\mu x-\epsilon),$$

or

$$\frac{S}{\mu}=-C\sin(\mu x-\epsilon) \quad . \quad . \quad (4)$$

The expression $m=C\cos(\mu x-\epsilon)$ in equation (3) can be represented graphically as follows —

Mark off OP at angle ϵ to OY, which represents the mid-point of the beam, and draw radial lines OI, OII . . . OA at angles μx_1, μx_2, μa to OY, i.e. at angles $(\mu x_1-\epsilon)$, $(\mu x_2-\epsilon)$ . . . $(\mu a-\epsilon)$ to OP (see Fig. 31).

From P drop perpendiculars $P(1)$, $P(2)$ . . . $P(a)$ on to these radial lines.

Then

$$O(1) = OP \cos(\mu x_1 - \epsilon) = m_1,$$
$$O(2) = OP \cos(\mu x_2 - \epsilon) = m_2,$$
$$O(a) = OP \cos(\mu a - \epsilon) = m_a.$$

The locus of P, (1), (2) . . . (a) is a semicircle with $OP = m$ as diameter, since angles $P(1)O$, $P(2)O$, . . . $P(a)O$ are right angles, and angles in a semicircle are right angles. In other words, OP is the maximum ordinate, m

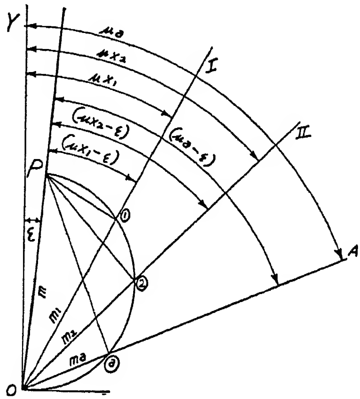


FIG. 31.—Graphical representation of $m = C \cos(\mu x - \epsilon)$

Considering the procedure in reverse, given μ, a and m_a, we can draw OA at an angle μa to OY, mark off m_a and so find OP and ϵ.

This is the essence of a Howard diagram, examples of which follow.

OY represents the origin or mid-point of the strut and OA the point of application of the end load at A, so that

$$m_a = \text{net B.M. at the end} = M_A - \frac{w}{\mu_2},$$

where

$$M_A = \text{Fixing Moment at } A$$

Example 14.—Strut with End Load and End-fixing Moments.

GIVEN: End load $P = 745$ lb. (see Fig. 32).

No distributed load, i.e. $w = 0$.

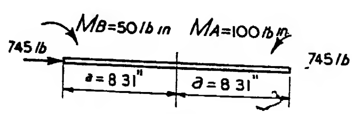


FIG. 32.

Applied Fixing Moment—

$$M_A = 100 \text{ lb. in.} \qquad E = 10{\cdot}5 \times 10^6$$
$$M_B = \ 50 \text{ lb. in.} \qquad I = {\cdot}005 \text{ in.}^4$$
$$a = 8{\cdot}31 \text{ in.} \qquad EI = {\cdot}0525 \times 10^6$$

$$\mu^2 = \frac{P}{EI} = \frac{745 \times 10^{-6}}{{\cdot}0525} = {\cdot}0142.$$

$$\mu = {\cdot}119.$$
$$\mu a = {\cdot}119 \times 8{\cdot}31 = {\cdot}99 \text{ radians.}$$
$$= 56{\cdot}75 \text{ deg.}$$

$$m = M - \frac{w}{\mu^2} = M, \quad \text{since} \quad w = 0.$$

Hence

$$= M_B = \ 50 \text{ lb. in.}$$
$$m_a = M_A = 100 \text{ lb. in.}$$

Procedure.—Draw OA and OB at angle μa to OY and mark off m_a and $m_b = 100$ and 50 respectively (see Fig. 33). Draw right angles at a and b.

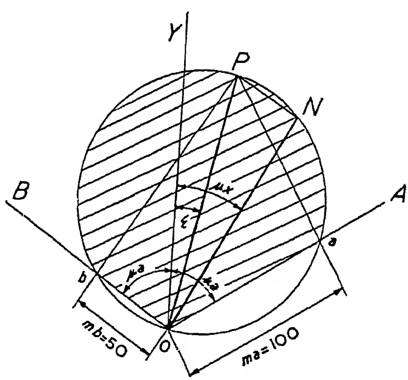


FIG 33.—The Howard diagram

Their point of intersection is P, and OP is the diameter of the circle which can be drawn through O, a, P and b. The bending moment diagram is shown shaded.

OP = maximum value of m at angle ϵ from OY, the mid-point of strut.

At any section distant x from centre of the strut, represented by an angle μx on the diagram,

Bending moment $= ON = OP \cos(\mu x - \epsilon)$, as in equation (3) above.

Similarly for *Shear*—

$$PN = C \sin(\mu x - \epsilon) = \frac{S}{\mu} \qquad \text{(cf. equation (4)).}$$

The general procedure may be stated thus—

(1) Calculate μa.
(2) Calculate w/μ^2 and thus m_a and m_b
(3) Draw OA and OB at μa to OY.
(4) Mark off m_a and m_b and draw perpendiculars to find P.
(5) Draw the semicircle on OP as diameter.
(6) Read off B.M. where required.

In the special case when $M_A = M_B$, maximum B.M. is given at once by $OP = \frac{m_a}{\cos \mu a}$, without drawing the Howard diagram.

Thus, in the above example, if $M_A = M_B = 100$ lb. in. (see Fig. 34),

$$\text{Maximum B.M. is at the centre and is } OP = \frac{100}{\cos 56{\cdot}75^\circ} \quad \frac{100}{\cdot 5483}$$
$$= 182 \text{ lb. in.}$$

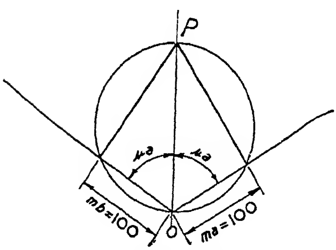


FIG. 34.—When $M_A = M_B$, the bending moment is given without drawing the Howard diagram.

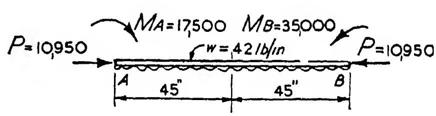


FIG 35.—A strut with end load, distributed load and end-fixing moments.

Howard Diagram for Strut with End Load, Distributed Load and End-fixing Moments.

Case (*a*).—Both end-fixing moments of the same sign and opposing the bending due to the distributed load (see Fig 35).

$$\mu^2 = \frac{P}{EI} = \frac{10{,}950}{22 \times 10^6} = 4{\cdot}97 \times 10^{-4}.$$

$$w = 42 \text{ lb./in.}$$

$$\left.\begin{aligned} E &= 30{\cdot}5 \times 10^6 \\ I &= 72 \end{aligned}\right\} EI = 22 \times 10^6$$

$$\mu = 2{\cdot}23 \times 10^{-2}.$$

(1) $\mu a = 2{\cdot}23 \times 10^{-2} \times 45 = 1{\cdot}00$ rad. $= 57{\cdot}3$ deg

(2) $\dfrac{w}{\mu^2} = \dfrac{}{4{\cdot}97 \times 10^{-4}} = 84{,}400$ lb. in.

$$m_a = M_A - \frac{w}{\mu^2} = 17{,}500 - 84{,}400 = \underline{-66{,}900 \text{ lb. in.}}$$

$$m_b = 35{,}000 - 84{,}400 = \underline{-49{,}400 \text{ lb. in.}}$$

Note.—It is usual for w/μ^2 to be numerically $> M_A$ or M_B, as here.

Since m_a and m_b are negative, draw OA and OB downwards at an angle μa to OY and mark off $O_a = m_a$ and $O_b = m_b$ (Fig. 36). Draw a circle of radius

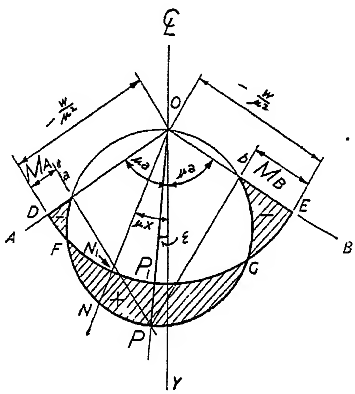


FIG 36.—Howard diagram for the strut shown in Fig 35.

w/μ^2 (w/μ^2 is constant throughout the span for constant I) with O as centre, and construct right-angles at a and b to meet at P Draw circle $OaPb$

Then

Maximum B.M. is at P and $= PP'$,

and

B.M. at distance x from the centre $= NN'$.

The B.M. diagram is shown shaded, points of contraflexure being at F and G.

Case (*b*)—As (*a*), with end moments greater, but still less than w/μ^2 (see Fig. 37).

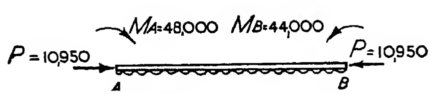


FIG. 37.—A strut similar to that in Fig. 35, but with end moments greater.

$$m_b = M_B - \frac{w}{\mu^2}$$
$$= 44{,}000 - 84{,}400$$
$$= \underline{-40{,}400 \text{ lb. in.}}$$
$$m_a = 48{,}000 - 84{,}000$$
$$= \underline{-36{,}400 \text{ lb. in.}}$$

The B.M. diagram in Fig 38 is shown shaded. It will be noticed that there is no point of contraflexure.

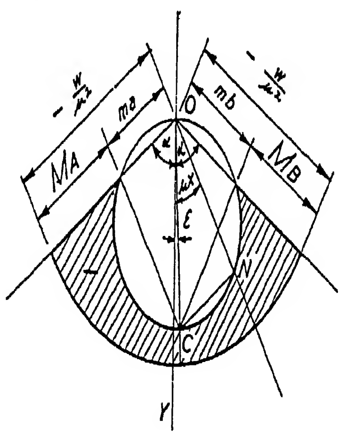


FIG. 38.—Howard diagram for the strut shown in Fig. 37.

The examples worked out above indicate the method of using Howard diagrams for end load, end-fixing, and distributed lateral load.

In cases where the lateral loads are concentrated, the diagrams become rather more involved, and will not be discussed here.

CHAPTER VI.

DISTRIBUTION OF SHEAR STRESS.

IN order to find the distribution of shear stress over a section, the following method can be employed, provided that the section is such that the distance of the centroid of any portion of it above the neutral axis is known (see Fig. 39).

It can be proved that the shear stress (lb /in.^{2}) at any section YY may be represented by—

$$q=\frac{FA\bar{y}}{I_{NA}\times b},$$

where

F = shear at section (lb.),
A = area above the section YY at which q is required (in.^{2}),
$\bar{y}$ = distance of C.G. of this area from neutral axis (in),
I_{NA} = M.I. of *complete* section about neutral axis (in ^{4}), and
b = thickness (in.).

The application of this formula will be demonstrated by worked examples.

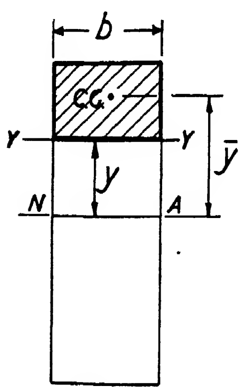


FIG. 39.—Diagram for finding the shear stress over a section.

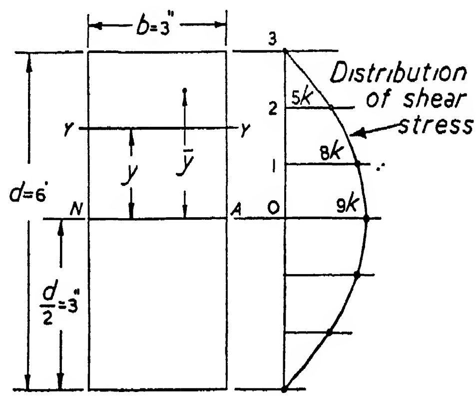


FIG. 40.—Distribution of shear stress on a rectangular section

Example 15.—Shear Stress on Rectangular Section.

Solid rectangular section (3 in. × 6 in.) as shown in Fig 40.

Take any section YY, y in. from the neutral axis.

$$\text{Area} \quad A = 3 \times (3 - y);$$

$$\bar{y} = y + \frac{3 - y}{2} = \frac{1}{2}(3 + y);$$

$$A\bar{y} = \frac{3}{2}(3 - y)(3 + y)$$

$$= \frac{3}{2}(9 - y^2).$$

$$I_{NA} = \frac{bd^3}{12} = \frac{3 \times 6^3}{12} = 54 \text{ in}^4$$

$$b = 3 \text{ in}$$

$$q = \frac{FA\bar{y}}{I_{NA}b} = \frac{F \cdot \frac{3}{2}(9 - y^2)}{54 \times 3} = \frac{(9 - y^2)F}{108}$$

$$= (9 - y^2)K, \quad \text{where } K \text{ is a constant} = \frac{F}{108}.$$

Taking various values of y, we can find values of q in terms of K and plot:

y (in.)	3	2	1	0
$(9 - y^2)K$	0	$5K$	$8K$	$9K$

The curve showing the distribution of shear stress is then as in Fig. 40.

$$\text{Mean shear stress} = \frac{F}{\text{Area}} = \frac{F}{18}.$$

$$\text{Maximum shear stress, which occurs when } y = 0, = \frac{9F}{108}$$

$$\text{Ratio } \frac{\text{Max. shear stress}}{\text{Mean shear stress}} = \frac{162}{108} = 1{\cdot}5.$$

Thus, for a rectangular section as shown, the maximum shear stress

$$= 1{\cdot}5 \times \text{mean shear stress},$$

$$= 1{\cdot}5 \frac{F}{bd} \text{ in general.}$$

Similarly, for a solid circular section, the maximum shear stress

$$=\frac{4}{3}\text{ mean shear stress,}$$

$$=\frac{4}{3}\frac{F}{\pi r^2},\quad\text{where } r=\text{the radius.}$$

Example 16.—To Find the Distribution of Shear Stress over an I-section.

(*a*) *Flange.*

Consider any section YY at a distance y from the neutral axis (Fig 41).

Area above YY—

$$A=1\times(\cdot80-y)$$
$$=(\cdot80-y).$$

$$\bar{y}=y+\frac{(\cdot80-y)}{2}=\frac{1}{2}(\cdot80+y).$$

$$A\bar{y}=\frac{1}{2}(\cdot80-y)(\cdot80+y)=\frac{1}{2}(\cdot64-y^2)$$

$$I_{NA}=\frac{bd^3}{12}=\frac{1\times1\cdot6^3}{12}=\underline{\cdot341\text{ in}^4}$$

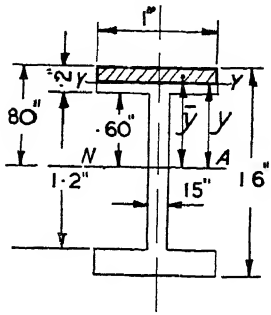


FIG 41 —Diagram for Example 16, which applies to the flange

$b=1$ in. for flange.

$$q=\frac{FA\bar{y}}{I_{NA}b}=\frac{F\times\frac{1}{2}(\cdot64-y^2)}{\cdot341\times1}$$

$$=\frac{F(\cdot64-y^2)}{\cdot682}=(\cdot64-y^2)K,$$

where $$K=\frac{F}{\cdot682}.$$

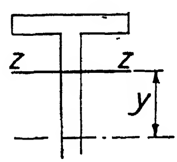


FIG 42 —Diagram for Example 16, which applies to the web

This expression holds for the flange only, i.e from $y=\cdot60$ in. to $y=\cdot80$ in. Stress distribution for flange is.

y (in)	60	70	·80
q	$28K$	$\cdot15K$	0
	$41F$	$22F$	0

(*b*) *Web.*

Total $A\bar{y}$ of section above ZZ in Fig 42 (note that we include the area of the flange, although only considering the shear stress in the web)—

$$A\bar{y} = A_1\bar{y}_1 + A_2\bar{y}_2$$

$$= (1 \times \cdot 2 \times \cdot 70) + \cdot 15\,(\cdot 60 - y)\left(y + \frac{\cdot 60 - y}{2}\right)$$

$$= \cdot 14 + \frac{\cdot 15}{2}\,(\cdot 60 - y)(\cdot 60 + y)$$

$$= \cdot 14 + \frac{\cdot 15}{2}\,(\cdot 36 - y^2).$$

$$= \frac{[\cdot 14 + 075\,(\cdot 36 - y^2)]\,F}{\cdot 341 \times \cdot 15}$$

$$= \frac{[\cdot 14 + \cdot 075\,(\cdot 36 - y^2)]\,F}{\cdot 051}$$

$$= [\cdot 14 + \cdot 075\,(\cdot 36 - y^2)]\,K,$$

where $$K = \frac{F}{\cdot 051}$$

Values of q for given values of y:—

y (in.)	0	20	40	·60
q	$167K$	$\cdot 164K$	$155K$	$14K$
	$3\ 27F$	$3\ 22F$	$3\ 04F$	$2\ 74F$

The "*Top-hat*" *Distribution of Shear Stress* is indicated in Fig. 43.

FIG. 43.—Diagram showing the "top-hat" distribution of shear stress for an I-section.

Note that the shear stress increases suddenly when passing from the flange to the web, and that most of the shear stress is in the web.

Example 17.—Symmetrical Box Spar Section.

Spruce flanges, ply webs.

(*a*) *Flange.*

At any section y (see Fig. 44),

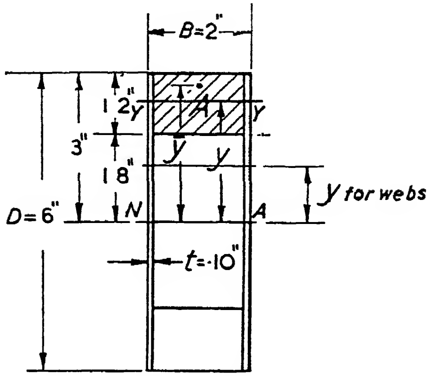


FIG. 44.—A symmetrical box spar section.

$$A = 2 \times (3 - y),$$

$$\bar{y} = y + \frac{3-y}{2} = \frac{3+y}{2};$$

$$A\bar{y} = \frac{2}{2}(3-y)(3+y),$$

$$= (9 - y^2).$$

$$q = \frac{FA\bar{y}}{I_{NA}b} = \frac{F(9-y^2)}{27 \times 2}$$

$$= \frac{F}{54}(9 - y^2).$$

t = flange thickness = 2 in.

$I = I_{NA}$ of whole section,

$$= \frac{2}{12}[6^3 - 3{\cdot}6^3]$$

$$= \frac{1}{6}[216 - 54]$$

$$= \frac{1}{6} \times 162 = \underline{27 \text{ in.}^4}$$

This is an approximate value, neglecting some of the web thickness, which in any case is very small.

This holds over the flange.

Values of q:—

y (in.)	3	2	1 8
$9 - y^2$	0	5	5 76
q	0	093F	·107F

(b) *Web.*

$$A\bar{y}=2\left\{\begin{array}{c}\overset{A_1}{(2\times1\cdot2)}\times\overset{\bar{y}_1}{2\cdot4}\\ +\ 10\times(1\cdot8-y)\left(\dfrac{1\cdot8-y}{2}+y\right)\end{array}\right\}$$

$$=2\left\{5\cdot78+\frac{\cdot10}{2}\,[3\cdot24-y^2]\right\}$$

Note.—There are two webs, so that $b=2t=\cdot20$ in

$$q=\frac{[11\cdot56+\cdot10\,(3\cdot24-y^2)]\,F}{27\times\cdot20}$$

$$=[11\cdot56+\cdot10\,(3\cdot24-y^2)]\,\frac{F}{5\cdot4}.$$

Values of q:—

y (in)	0	5	1 0	1 5	1 8
q	11 88	11 86	11 78	11 66	11 56
	2 2F	2 2F	2 18F	2·16F	2 14F

The distribution of shear stress is shown in Fig. 45.

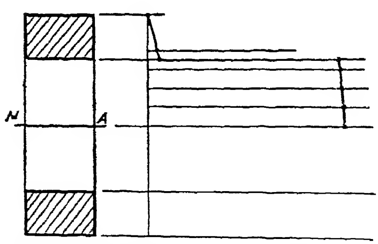


Fig. 45.—Diagram for stress distribution in Example 17.

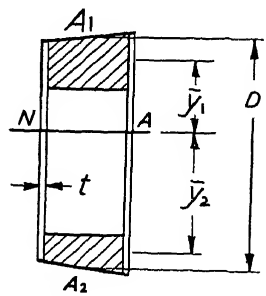


Fig. 46.—An unsymmetrical box spar section

Application.—An actual spar section is often unsymmetrical (because the top and bottom flanges are not of the same depth), but since the maximum shear stress in the web occurs at the neutral axis, the method is to take $A_1\bar{y}_1$, as shown in Fig 46 (or $A_2\bar{y}_2$, the product will be the same, for this is how the N.A. is found) Considering Fig. 47 and using mean dimensions,

$$A_1\bar{y}_1 = \cdot 709 \times 2\cdot 08 \times \frac{2\cdot 08}{2} - \cdot 63 \times 1\cdot 58 \times \frac{1\cdot 58}{2},$$

$$= \cdot 709 \times \frac{2\cdot 08^2}{2} - \cdot 63 \times \frac{1\cdot 58^2}{2},$$

$$= 1\cdot 54 - \cdot 787$$

$$= \cdot 75 \text{ in.}^3.$$

Then $$q = \frac{\cdot 75F}{I_{XA}b}, \quad \text{where } b = 2 \times t$$

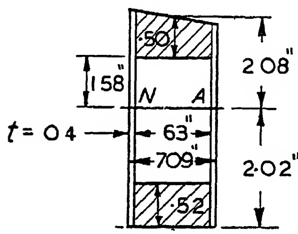


FIG. 47.—Mean dimensions of spar section.

Shear Stress Due to Pure Torsion.—The shear stress due to pure torsion for the most common sections is given in Table VII (see also Air Publication 970, VIII, III, 2),

where T = torque in lb. in.,
q = torque shear stress (lb /in.²).

Applications of these formulæ will be given in Part II under "Detail Stressing."

Torsional Deflection of a Tube.—The torsional deflection of a tube in a control system, etc. should not exceed 1 deg. per foot, the equation for torsional deflection being.

$$\frac{\theta}{l} = \frac{T}{GI_p} \text{ rad /in },$$

where

θ = angle of twist (radians),
l = length of tube (in.),
T = torque (lb. in.),
I_p = polar moment of inertia (in.^{4}),
G = Modulus of Rigidity = $12\cdot 5 \times 10^6$ for steel (T 45) tubes, and
$4\cdot 2 \times 10^6$ for duralumin (T.4) tubes.

TABLE VII.—SHEAR STRESS DUE TO PURE TORSION.

Type of Section	Formula for Torque Shear Stress	Position of Shear Stress.
Solid circle Diameter D.	$q=\frac{16T}{\pi D^3}$	At boundary
Hollow circle Outside Diameter D. Inside Diameter d	$q=\frac{16TD}{\pi(D^4-d^4)}$	At boundary.
If thickness t is small compared with D.	$q=\frac{2T}{\pi tD^2}$	At boundary
Solid ellipse Major Axis $2a$. Minor Axis $2b$.	$q=\frac{2T}{\pi ab^2}$ $=\frac{2T}{\pi a^2b}$	At end of minor axis At end of major axis
Any hollow section Thickness t small compared with smallest outside dimensions. A = Area bounded by mean perimeter.	Approx. $q=\frac{T}{2At}$	Any point on boundary where thickness $=t$. This is Batho's formula
Solid square Side b.	$q=4{\cdot}8\frac{T}{b^3}$	At middle of sides.
Solid rectangle Long side a. Short side b.	Approx. $q=\frac{T}{ab^2}\left(3+1{\cdot}8\frac{b}{a}\right)$	At middle of long side

Example 18.

Find the torsional deflection of a T.45 tube, $1\frac{1}{2}$ in. O/D × 17G., subjected to a torque of 2400 lb. in.

$$I_p = {\cdot}1326 \text{ in.}^4.$$

$$\frac{\theta}{l}=\frac{2400\times 12\times 57{\cdot}3}{12{\cdot}5\times 10^6\times{\cdot}1326}\text{ deg /ft.}$$

$$=\underline{{\cdot}995 \text{ deg./ft.}}$$

This is within the prescribed limit.

CHAPTER VII.

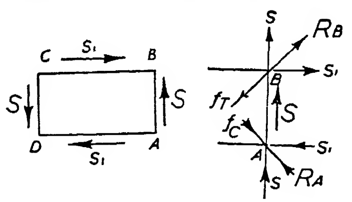
THIN-WEB BEAMS.

In ordinary structural engineering it is usual to consider that the shear at any section of a beam is taken as a shear stress in the web, which is of sufficient thickness to withstand such stress without buckling. In aircraft structures, however, the shear is usually small compared with the depth of the beam, and a thin web is sufficient to carry it. The critical stress of a thin web is, however, low, and folds will begin to form long before the ultimate shear stress of the material is reached

By riveting vertical stiffeners to the web between the booms, and so reducing the size of the panel, the critical stress will be increased, and, if the spacing is sufficiently small, will equal or even exceed the allowable stress in pure shear.

This is uneconomical, however, and provided that the distance between stiffeners is not more than half the depth of the beam, it is quite safe to allow folds to form in the web, which will then carry the shear, not as a shear stress, but as a tensile stress in the direction of the folds. This theory was evolved by Professor Wagner, and the folds so developed are known as Diagonal Tension Fields.

Diagonal Tension Fields.—If we apply a shear S to the faces of an infinitely small rectangular block $ABCD$, we know that, for equilibrium, there must be an equal and opposite couple constituted by the shears S_1 acting along AD and CB, as shown in Fig. 48.



FIGS. 48 and 49 —(Left) shear applied to the face of a small rectangular block, and (right) the resultant

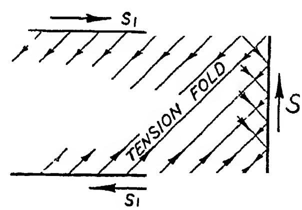


FIG. 50 —When the critical stress is exceeded, the compressive stresses buckle the web into folds.

Thus at B we have a resultant R_B giving a tensile stress f_T, and at A a resultant R_A giving a compressive stress f_C (Fig. 49).

When the critical stress is exceeded, the compressive stresses have the effect of buckling the web into folds along the lines of f_T, a fact which can be demonstrated by laying a piece of paper flat on the table and applying a shear load at one end whilst holding the other, whereupon the folds will be seen (see Fig. 50).

Consequently, any further increase in the shear can only be taken by an increase in the tensile stress f_T.

It may also be assumed that the shear taken up to the critical stress of the panel is carried in pure shear beyond that point, and only the remainder of the shear is taken by tension fields. Hence, the shear that gives the critical stress of the panel should be subtracted from the total shear, and this figure used in determining the tensile stress in the web

The critical stress (f_{CR}) of a flat plate, simply supported, subject to shear, is given by the formula

$$f_{CR} = 4{\cdot}8E\left(\frac{t}{b}\right)^2 + 3{\cdot}6E\left(\frac{t}{a}\right)^2 \text{ lb /in.}^2,$$

where

E = Young's modulus (lb./in.^{2}),
t = thickness of panel (in.),
a = maximum dimension of panel (in), and
b = minimum dimension of panel (in.).

This is only one of many formulæ that may be used

For a square panel, the direction of the folds will be at 45 deg to the applied shear, and even for rectangular shapes the angle does not depart far from this, provided that the edge members are stiff

It will be clear that the tension in the web will tend to pull the booms towards each other, and it is therefore necessary to arrange for flange spacers (or stiffeners) at frequent intervals to counteract this effect. These stiffeners, as explained before, also reduce the critical stress in the web.

Factor governing the choice of a Thin-web Beam:

If S = total shear (lb.), and
h = depth of beam (in.) between the neutral axes of booms,

the index value $= \dfrac{\sqrt{S}}{h}$; and if this value is less than 7, as frequently happens in aircraft, a thin-web beam is usually considered preferable.

Formulæ for Thin-web Beams.—Table VIII gives formulæ for use with thin-web beams with parallel or non-parallel flanges. The proofs are not given, but the worked examples following the Table show their application.

TABLE VIII.—FORMULÆ FOR THIN-WEB BEAMS.

	Parallel Flanges	Non-parallel Flanges
Shear in Web S_W	S	$S-\frac{M}{h}(\tan\delta T+\tan\delta C)$
Tensile Stress in Web f_W (lb /in.²)	$\frac{2S}{ht}$	$\frac{2S_W}{ht\times C_2}$
Horizontal Load in Flange Members (lb)	Compression: $H_C=\frac{M}{h}+\frac{S}{2}$ Tension. $H_T=\frac{M}{h}-\frac{S}{2}$	$\frac{M}{h}+\frac{S_W}{2}$ $\frac{M}{h}-\frac{S_W}{2}$
Vertical Component of Web Tension V (lb)	$\frac{Sd}{h}$	$\frac{S_Wd}{h}$
Flange Bending Moment M_{F^1} (lb in)	$\frac{Sd^2}{12h}\times C_1$	$\frac{S_Wd^2}{12h}\times C_1$
Stress in Booms (lb /in.²)	Tensile: $f_T=\frac{H_T}{A_T}+\frac{M_{F^1}\times y_T}{I_T}$ Compressive. $f_C=\frac{H_C}{A_C}+\frac{M_{F^1}\times y_C}{I_C}$	

Nomenclature

S = applied shear (lb).
h = distance between neutral axes of top and bottom booms (in.)
d = stiffener spacing (in).
M = bending moment (lb in).
t = thickness of web (in)
C_1 and C_2 = constants obtainable in any given instance, they depend on the shape of the panel
y_T and y_C distance (in) of N.A. to outermost fibre
A_T and A_C area (in ²)
I_T and I_C the Moments of Inertia (in ⁴)
δT and δC = the angles of divergence to the horizontal (degrees)
} of the tension and compression booms respectively.

Example 19.—Thin-web Beam with Parallel Flanges.

The shear at a section of a thin-web parallel-flange cantilever beam is 2870 lb. upward, the allowable shear in buckling in the web being 2290 lb. The B.M. is 191,150 lb. in. If the depth between the neutral axes of top and bottom booms (h) is 6·37 in., stiffener spacing (d) = 4·5 in. and $C_1 = 0{\cdot}97$, find H_C, H_T, M_{F^1}, f_C and f_T in the booms (see Fig. 51).

I of each boom (L.40) = ·0992 in ⁴;
y_C = ·614 in.; y_T = ·586 in.
$A_C = A_T$ = ·665 in ².

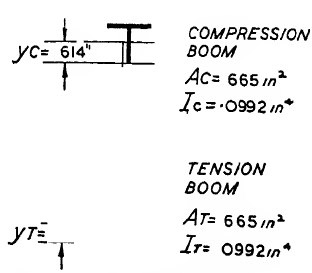


FIG. 51 —Diagram for Example 19.

Shear taken by web in tension folds $= 2870 - 2290$
$= 580$ lb.

$$H_C = \frac{M}{h} + \frac{S}{2} = \frac{191{,}150}{6{\cdot}37} + \frac{580}{2} = 30{,}000 + 300 = 30{,}300 \text{ lb.}$$

$$H_T = 30{,}000 - 300 = 29{,}700 \text{ lb.}$$

$$M_{F^1} = \frac{Sd^2}{12h} \times C_1 = \frac{580 \times 4{\cdot}5^2}{12 \times 6{\cdot}37} \times {\cdot}97 = 149 \text{ lb. in.}$$

$$f_C = \frac{H_C}{A_C} + \frac{M_{F^1} \times y_C}{I_C} = \frac{30{,}300}{{\cdot}665} + \frac{149 \times {\cdot}614}{{\cdot}0992} = 45{,}600 + 920 = 46{,}520 \text{ lb./in.}^2$$

L 40 at 47,000 lb./in.² Reserve Factor (R.F) $= 1{\cdot}01$.

$$f_T \quad \frac{29{,}700}{{\cdot}665} \quad M_{F^1} \times y_T = 44{,}700 + \frac{149 \times {\cdot}586}{{\cdot}0992} = 44{,}700 + 880 = 45{,}580 \text{ lb./in.}^2$$

L.40 at 56,000 lb./in.² R.F. $= 1{\cdot}22$.

Due to upward shear, the beam will bend upwards as shown in Fig. 52, putting M_1N_1 in compression and MN in tension, but due to diagonal tension in the web the booms will tend to take the shape MQN and $M_1Q_1N_1$,

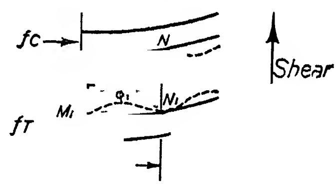


FIG. 52.—Due to upward shear, the beam will bend as shown

giving compression at M and M_1, and tension at Q and Q_1. Hence, maximum compression is at M_1 and maximum tension at Q, giving us the appropriate values of y_C and y_T.

Example 20.—Thin-web Beam with Non-parallel Flanges.

Given:

$S = 1190$ lb.	$d = 6$ in.
$M = 83{,}000$ lb. in.	$I_T = {\cdot}03334$
$h = 5{\cdot}877$ in.	$A_T = {\cdot}2786$
$\tan \delta_T = {\cdot}01067$	$I_C = 0481$
$\tan \delta_C = {\cdot}01532$	$A_C = {\cdot}3574$
$C_1 = {\cdot}915$	$y_T = {\cdot}872$
$C_2 = {\cdot}74$	$y_C = {\cdot}343$.

$$(\tan \delta_T + \tan \delta_C) = {\cdot}01067 + {\cdot}01532 = {\cdot}026.$$

Referring to Table VIII,

$$S_W = S - \frac{M}{h}(\tan \delta_T + \tan \delta_C) = 1190 - \frac{83{,}000}{5{\cdot}877} \times 026 = 823 \text{ lb.}$$

$$H_T = \frac{M}{h} - \frac{S_W}{2} = \frac{83{,}000}{5{\cdot}877} - \frac{823}{2} = 13{,}719 \text{ lb.}$$

$$H_C = \frac{M}{h} + \frac{S_W}{2} = 14{,}542 \text{ lb.}$$

$$M_{F^1} = {\cdot}915 \times \frac{S_W d^2}{12h} = \frac{{\cdot}915 \times 823 \times 36}{12 \times 5{\cdot}877} = 384 \text{ lb. in.}$$

$$f_T = \frac{H_T}{A_T} + \frac{M_{F^1} \times y_T}{I_T} - \frac{13{,}719}{2786} + \frac{384 + {\cdot}872}{{\cdot}03334} = 49{,}300 + 10{,}050 = 59{,}350 \text{ lb./in.}^2$$

$$f_C = \frac{14{,}542}{3574} + \frac{384 \times {\cdot}343}{{\cdot}0481} \quad 43{,}440 \text{ lb./in.}^2$$

$$f_W \text{ (max.)} = \frac{2S_W}{ht \times C_2} = \frac{2 \times 823}{5{\cdot}877 \times {\cdot}028 \times {\cdot}74} = 13{,}500 \text{ lb /in.}$$

$$V = \frac{S_W d}{h} = \frac{823 \times 6}{5{\cdot}877} = 840 \text{ lb.}$$

CHAPTER VIII.

FRAMEWORKS.

IN aircraft frameworks such as engine mountings, undercarriage structures, etc., most of the members lie in more than two planes, and the finding of the loads in them is facilitated by making use of Direction Cosines. An actual stress diagram is rarely drawn, except perhaps as a check on some particular joint.

In any framework, let OP be a member of length l whose perpendicular distances from the planes YZ, XZ and XY, mutually at right angles, are x, y and z respectively (see Fig. 53).

From the triangle OAP,

$$OP^2 = OA^2 + AP^2 = OA^2 + OD^2.$$

But, from triangle OAB,

$$OA^2 = BA^2 + OB^2$$
$$= OC^2 + OB^2,$$

i.e. $$OP^2 = OB^2 + OC^2 + OD^2$$

or $$l^2 = x^2 + y^2 + z^2$$

and $$l = \sqrt{x^2 + y^2 + z^2}.$$

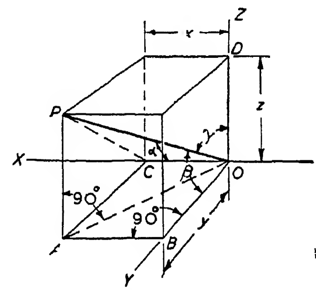


FIG 53 —Diagram for determining the direction cosines

If α, β and γ are the angles made by OP with the axes X, Y and Z respectively,

$$\cos\alpha = \frac{OC}{OP} = \frac{x}{l},$$

$$\cos\beta = \frac{OB}{OP} = \frac{y}{l},$$

and $$\cos\gamma = \frac{OD}{OP} = \frac{z}{l}.$$

The values $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the Direction Cosines of OP, and are such that:

The projection of OP on the X axis $= OP\cos\alpha$.
" " " Y axis $= OP\cos\beta$.
" " " Z axis $= OP\cos\gamma$.

In any given case, therefore, we can, from the geometry of the structure, find x, y and z, then l and the direction cosines, and by considering each joint in turn, equate the external X, Y and Z loads to the projections of the members at that joint on the X, Y and Z axes, and so find the load in each member.

First, the method of tabulation for direction cosines will be shown by a worked example, after which a typical engine-mounting structure will be studied.

Example 21.—Direction Cosines.

Figs. 54 and 55 show elevation, plan and pictorial views of a structure acted upon by the following external loads:—

X	load	(+forward)	$=200$ lb.
Y	,,	(+to starboard)	$=100$ lb.
Z	,,	(+upward)	$=-300$ lb.

(This is the usual convention for X, Y and Z loads.)

We now wish to find the load in each member of the structure.

Procedure.—Set out in tabular form the values x, y and z for each member AB, AC and AD, and so find l and the direction cosines (Table IX). A check on the working is afforded by squaring the direction cosines of any member and adding. The result should be unity, since

$$\left(\frac{x}{l}\right)^2+\left(\frac{y}{l}\right)^2+\left(\frac{z}{l}\right)^2=\frac{x^2+y^2+z^2}{l^2}=\frac{l^2}{l^2}=1.$$

TABLE IX —DIRECTION COSINES (EXAMPLE 21).

									$\cos\alpha$	$\cos\beta$	$\cos\gamma$	
Member.	x.	y.	z	x^2	y^2.	z^2.	l^2.	l.	x/l	y/l.	z/l	Check
AB	25	5	4 5	625	25	20 3	670	**25·9**	965	193	174	997
AC	25	13 5	4 5	625	182	20 3	827	**28·7**	872	471	157	1 007
AD	25	.	14·5	625		210	835	**28·9**	865	.	502	1 002

(a) *First consider "X" loads at A.*

Since we do not know the direction of the loads in AB, AC and AD, assume that they are in tension. That is, their projections on the X plane

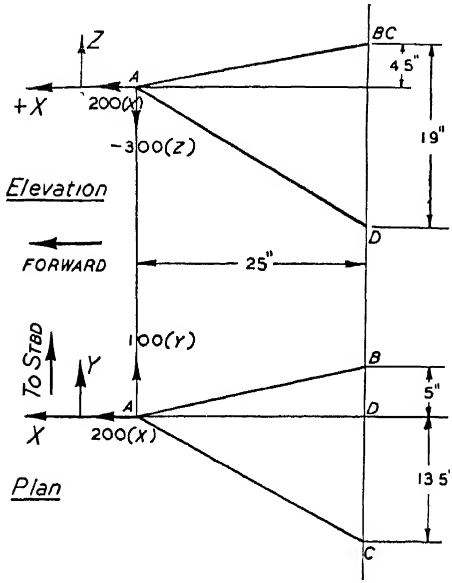


FIG. 54 —Plan and elevation of structure acted upon by external loads.

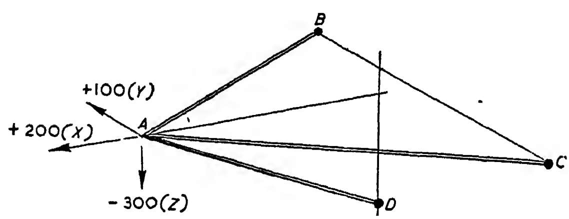


FIG. 55.—Pictorial view of structure.

will, in this case, all be in the negative direction (see Fig. 56), the equation of equilibrium at A being—

$$\text{‘}X\text{’'}:\ 200 - AB\cos\alpha_{AB} - AC\cos\alpha_{AC} - AD\cos\alpha_{AD} = 0$$

$$200 - AB \times \cdot965 - AC \times \cdot872 - AD \times \cdot865 = 0$$

$$\cdot965AB + \cdot872AC + \cdot865AD = 200 \qquad (1)$$

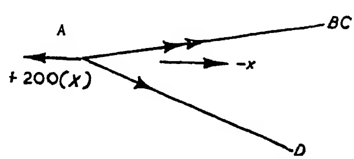


FIG. 56.—When AB, AC and AD are in tension, the projections of the loads on the X plane will all be in the negative direction.

(*b*) "*Y*" *loads at A.*

Still assuming tension in each member (Fig. 57),

The projection of AB is $+$,
,, ,, ,, AC is $-$,
,, ,, ,, AD is nil,

since AD has no Y direction cosine.

$$\text{"}Y\text{"}: \; 100 + AB \cos \beta_{AB} - AC \cos \beta_{AC} = 0.$$

$$\cdot 193 AB - \cdot 471 AC = -100 \quad . \qquad (2)$$

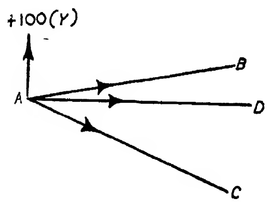


Fig. 57.—The projection of AB is positive, of AC is negative, and of AD is nil.

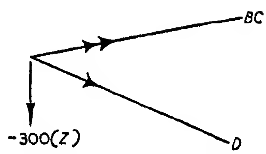


Fig 58 —Determination of Z loads at A.

(*c*) "*Z*" *loads at A* (Fig. 58).

Projections of AB and AC are $+$,
Projection of AD is $-$.

$$\text{"}Z\text{"}: \; -300 + AB \cos \gamma_{AB} + AC \cos \gamma_{AC} - AD \cos \gamma_{AD} = 0$$

$$\cdot 174 AB + \cdot 157 AC - \cdot 502 AD = 300 \quad . \quad . \qquad (3)$$

We have three equations and can therefore solve for the three unknowns.

From (2):

$$\cdot 193 AB = -100 + \cdot 471 AC$$

$$AB = -518 + 2 \cdot 44 AC \quad . \quad . \qquad (2a)$$

Substituting in (1):

$$\cdot 965 \, [2 \cdot 44 AC - 518] + \cdot 872 AC + \cdot 865 AD = 200$$

$$2 \cdot 35 AC - 500 + \cdot 872 AC + \cdot 865 AD = 200$$

$$3 \cdot 222 AC + \cdot 865 AD = 700 \quad . \quad . \quad . \qquad (4)$$

Substituting in (3):

$$\cdot 174 \, [2 \cdot 44 AC - 518] + \cdot 157 AC - \cdot 502 AD = 300$$

$$\cdot 425 AC - 90 + \cdot 157 AC - \cdot 502 AD = 300$$

$$\cdot 582 AC - \cdot 502 AD = 390 \quad . \qquad (5)$$

$$3 \cdot 222 AC - 2 \cdot 78 AD = 2160 \quad . \quad . \quad . \qquad (5a)$$

$$3 \cdot 222 AC + \cdot 865 AD = 700 \quad . \quad . \quad . \qquad (4)$$

Subtracting, $\qquad -3 \cdot 645 AD = 1460.$

$$\therefore \ AD = \underline{-400 \text{ lb.}} \text{ (Compression, since negative).}$$

From (5*a*)

$$3{\cdot}222AC - 2{\cdot}78 \times (-400) = 2160$$
$$3{\cdot}222AC = 2160 - 1113 = 1047$$
$$AC = \underline{324 \text{ lb}} \text{ (Tension).}$$

From (2*a*):

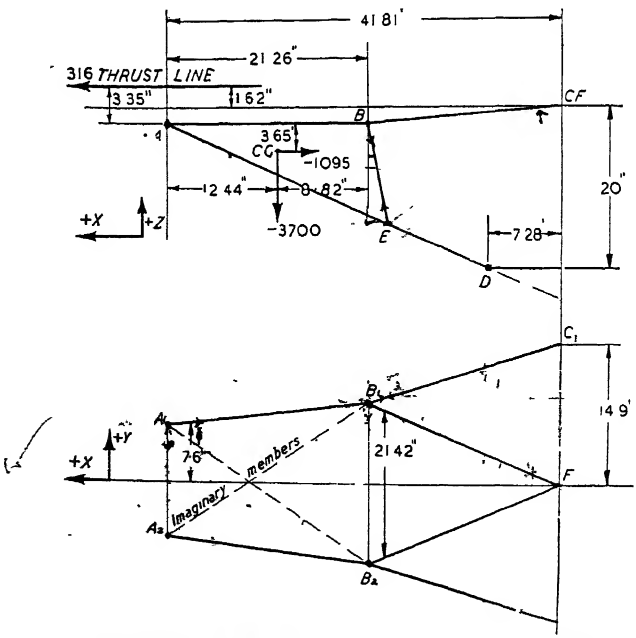
$$AB = -518 + 2{\cdot}44AC$$
$$= -518 + 2{\cdot}44 \times 324$$
$$= -518 + 790$$
$$= 272 \text{ lb. (Tension).}$$

Check.

From (1):

$$\text{L.H.S.} = {\cdot}965 \times 272 + {\cdot}872 \times 324 - {\cdot}865 \times 400$$
$$= 262 + 283 - 346$$
$$= 199, \text{ whereas R.H.S.} = 200.$$

Engine Mounting.—We now pass on to the stressing of the typical engine-mounting structure shown in Figs. 59 and 60.



FIGS. 59 (above) and 60.—Loads on a typical engine mounting

In Fig. 60 the engine-bearer feet are at A_1, A_2, B_1 and B_2, and the following is the notation:—

A_1B_2, A_2B_1, A_1A_2, B_1B_2 } Imaginary members—used for taking side load only. These loads will actually be taken by the engine

B_1F, B_2F } For taking side load only.

The engine is a left-hand tractor; that is, the airscrew revolves in a contra-clockwise direction looking from the rear of the machine. Having found the direction cosines of the members, the next step is to apply a unit X load of 100 lb. at A_1 and A_2, and trace its effect through the structure. Unit Y and Z loads of 100 lb. are then placed independently at A_1 and A_2, and the loads in all members found.

TABLE X.—DIRECTION COSINES FOR THE ENGINE MOUNTING.

Member	x.	y	z	x^2	y^2.	z^2.	l^2	l.	x/l.	y/l	z/l.	Check
AB	21 26	3 11	..	453	9 7	.	463	21 5	991	145	..	1·003
AE	22 51	3 25	12	508	10 6	144	663	25 7	876	126	467	1 000
BE	1·25	- 14	12	1 6	.	144	146	12 1	103	012	992	996
BC	20 55	4 19	1 73	423	17 6	3	444	21·0	98	20	082	1 000
BF	20 55	10 71	1 73	423	115	3	541	23 3	882	46	074	996
EC	19·30	4 05	13 73	373	16 4	189	578	24 0	804	169	·572	1 002
ED	12 02	2 5	6 27	145	6 3	39 4	191	13 8	872	182	·454	999
A_1B_2, A_2B_1	21·26	18 31		453	335	..	788	28 0	76	654		1 003

After this, unit X, Y and Z loads are placed independently at B_1 and B_2 (these loads will, of course, affect only the structure aft of these points) and the results tabulated. This has been done in Table XI.

TABLE XI.—SUMMARY OF LOADS IN MEMBERS FROM UNIT LOADING.

Member.	Loads at A_1 and A_2			Loads at B_1 and B_2		
	100X	100Y.	100Z	100X.	100Y	100Z.
A_1E_1	.	.	214 (T)	..		
A_1B_1	101 (T)	101 (C)	189 (C)	..	.	
A_1A_2	14 6 (T)	.	..	..		.
A_1B_2	..	131 (T)	.	..		
B_1C_1	102 (T)	286 (C)	191 (C)	101 (T)	141 (C)	10 5 (C)
B_1E_1	8 37 (T)	16 8 (C)	15 8 (C)	8 38 (T)	.	100 (T)
B_1B_2	5·7 (T)		11 (C)	20 3 (T)	..	..
B_1F	..	92 8 (T)		..	157 (T)	..
E_1D_1	10 (T)	18 (C)	201 (T)	10 (T)		98 8 (T)
E_1C_1	10 (C)	18 (T)	13 (T)	10 (C)		95 (C)

The procedure now is to find the actual loads applied to the engine-bearer feet as a result of gravity, thrust, torque, pitching, yawing, side load,

inverted flight, and static thrust and torque (cf. Air Publication 970, II, 18). These are expressed as X, Y and Z loads at A_1, A_2, B_1 and B_2, and by scaling from the values obtained from unit loading, we can find the *actual* loads in the members and therefore the maximum tension or compression in any particular case.

Stressing Case : Turning in Flight with Engine On.

(See Air Publication 970, II, 18.)

(*a*) *Gravity Forces:* Taking a factor of $N=9{\cdot}0$,

$$9\times W=9\times 428=3860 \text{ lb.},$$

where W = weight of engine, airscrew, cowling, etc = 428 lb.

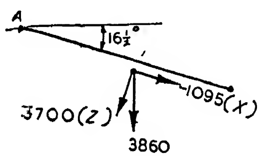


FIG. 61.—Resolution of gravity load.

With a stalling attitude of 16½ deg., the gravity load of 3860 lb. can be resolved into (see Fig 61)—

Load at right angles to thrust line $=-3860\cos 16\tfrac{1}{2}=\underline{-3700 \text{ lb. } (Z)}$.

Load parallel to thrust line $=-3860\sin 16\tfrac{1}{2}=\underline{-1095 \text{ lb } (X)}$

(*b*) *Airscrew Thrust and Torque :*

$$2\times\text{Thrust}=\frac{2\eta\times 550\times\text{H.P.}}{V_s\sqrt{\dfrac{N}{2}}}\text{ lb.},$$

where V_s = stalling speed = 82 ft./sec.

$$\therefore\quad V_s\sqrt{\frac{N}{2}}=82\sqrt{4{\cdot}5}=174 \text{ ft./sec.}=119 \text{ m.p.h.}$$

η = airscrew efficiency = 0·75.

H.P. = 67 at 119 m.p.h.

(It is assumed here that we know the above values from performance data.)

$$\therefore\ 2\times\text{Thrust}=\frac{2\times{\cdot}75\times 550\times 67}{174}=\underline{316 \text{ lb. } (X)}.$$

$$2 \times \text{Torque} = \frac{2 \times 33{,}000 \times \text{H.P.}}{2\pi \times \text{r.p.m.}}, \quad \text{where r.p.m of airscrew} = 2000;$$

$$= \frac{66{,}000 \times 67}{2\pi \times 2000} = 352 \text{ lb. ft.}$$

(c) *Gyroscopic Couple due to a Banked Turn without Side-slipping at a Speed* $V_s\sqrt{\frac{N}{2}}$:

$$2 \times \text{Gyroscopic Pitching Couple}: \ 2C_2 = \frac{2 \times I_P \,\Omega \sin \Phi}{V_{s\,}\sqrt{\frac{N}{2}}}$$

$$\cos \Phi = \frac{\Delta}{N}, \quad \text{where } \Phi = \text{angle of bank},$$

$$= \cdot 222. \qquad \Phi = 77^\circ\ 10', \quad \sin \Phi = \cdot 975, \quad \tan \Phi = 4{\cdot}3891.$$

I_P for two-bladed metal airscrew

$$= \cdot 018\, D^{4\cdot 4} \times 2 \text{ lb. ft.}^2.$$

For $D = 7$ ft.

$$I_P = 188{\cdot}2 \text{ lb. ft.}^2.$$

Ω = angular velocity of airscrew (radians per sec.)

$$= \frac{2000}{60} \times 2\pi = 209 \text{ radians per sec.}$$

$$\therefore\ 2C_2 = \frac{2 \times 188{\cdot}2 \times 209 \times \cdot 975}{174} = \underline{441 \text{ lb. ft.}}$$

2 × Gyroscopic Yawing Couple :

$$2C_1 = 2C_2 \tan \Phi = 441 \times 4{\cdot}3891 = \underline{1932 \text{ lb. ft.}}$$

Loads at Engine Bearers.

(1) −3700 lb. (Z) at C.G. gives—

At A_1 and A_2: $\quad \frac{1}{2} \times \frac{3700 \times 8{\cdot}82}{21{\cdot}26} = -767$ lb. (Z);

At B_1 and B_2: $\quad -1083$ lb. (Z).

(2) −1095 lb. (X) at C G. gives—

Direct Load at A_1, A_2, B_1 and $B_2 = \frac{-1095}{4} = -274$ lb. (X).

Due to offset: Load at A_1 and A_2 $\frac{1095 \times 3{\cdot}65}{2 \times 21{\cdot}26} = -94$ lb. (Z).

Load at B_1 and $B_2 = +94$ lb. (Z).

(3) *Thrust:* $+316$ (X) gives—

Due to offset. Load at A_1 and $A_2 = \dfrac{316 \times 3\cdot35}{2 \times 21\cdot26} = -25$ lb. (Z).

Load at B_1 and $B_2 = +25$ lb. (Z).

Direct load at A_1, A_2, B_1 and $B_2 = \dfrac{316}{4} = +79$ lb. (X).

(4) *Pitching Couple* (for a left-hand tractor a left turn always results in a pitching-down couple and a right turn in a pitching-up) $= 441 \times 12 = 5300$ lb. in., giving—

$$\text{Load at } A_1 \text{ and } A_2 = \frac{5300}{2 \times 21\cdot26}$$

$= -125$ lb. (Z) Left turn.
$+125$ lb. (Z) Right turn.

Load at B_1 and $B_2 = +125$ lb. (Z) Left turn.
$= -125$ lb. (Z) Right turn.

(5) *Yawing Couple* $= 1932 \times 12 = 23{,}200$ lb. in.

For a L.H. tractor, yaw is to the left, as shown in Fig. 62, for either a left or right turn.

$$\text{Load at } A_1 \text{ and } A_2 = \frac{23{,}200}{21\cdot26} = 1090 \text{ lb.}$$

$= -545$ lb (Y) at A_1 and A_2.
$= +545$ lb (Y) at B_1 and B_2.

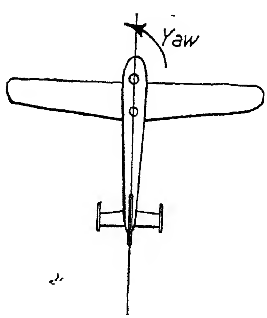


FIG. 62.—Showing direction of yaw.

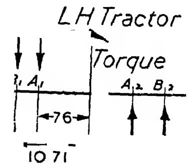


FIG 63 —Loads due to torque. (View looking aft)

(6) *Torque* (see Fig. 63) $= 352 \times 12 = 4224$ lb. in

The method used below for torque loads will be understood from a later section on eccentric loading on a rivet or bolt group (see pp. 92 *et seq.*).

Station.	r (in.)	r^2	$P=\frac{Mr}{\Sigma r^2}=\frac{4224r}{345\ 6}=12\ 2r$.
A_1	7 6	57 8	− 93 (Z)
A_2	7 6	,,	+ 93 (Z)
B_1	10 71	115 0	−131 (Z)
B_2	10 71	,,	+131 (Z)

$\Sigma r^2 = 345\ 6$

Sideload Case (see Air Publication 970, II, 19)

Unit gravity load $= \pm 428$ lb. (Y).

Direct Load:

$$\text{At } A_1 \text{ and } A_2 = \tfrac{1}{2} \times \frac{428 \times 8{\cdot}82}{21\ 26} = \frac{178}{2} = \underline{\pm 89 \text{ lb. } (Y).}$$

$$\text{At } B_1 \text{ and } B_2 = \frac{250}{2} = \underline{\pm 125 \text{ lb. } (Y).}$$

Table XII.—Summary of Loads at A_1, A_2, B_1 and B_2—Turning in Flight, Engine On.

		A_1			A_2			B_1.			B_2	
	X	Y.	Z.	X	Y.	Z	X	Y	Z.	X	Y.	Z
Gravity Load (1)			− 767			−767			−1083			−1083
Gravity Load (2)	−274		− 94	−274		− 94	−274		+ 94	−274		+ 94
Thrust . . .	+ 79		− 25	+ 79		− 25	+ 79		+ 25	+ 79		+ 25
Pitching Left Turn			− 125			−125			+ 125			+ 125
Pitching Right Turn			+ 125			+125			− 125			− 125
Yawing . .		−545			−545			+545			+545	
Torque. . .			− 93			+ 93			− 131			+ 131
	−195	−545	−1104	−195	−545	−918	−195	+545	− 970	−195	+545	− 708

Check $\Sigma X = -\ 780$ Applied $X = -1095 + 316 = -779$ lb
$\Sigma Y = 0$,, $Y = 0$
$\Sigma Z = -3700$,, $Z = -3700$ lb

Inverted flight (Load Factor = 4·5):

In this case the weight of the engine is assumed to act at right angles to the line through the engine-bearers.

$$\text{Up load on bearer feet} = 4{\cdot}5 \times 428$$
$$= +1930 \text{ lb. } (Z).$$

$$\text{At } A_1 \text{ and } A_2: \frac{1930}{2} \times \frac{8{\cdot}82}{21{\cdot}26} = +400 \text{ lb. } (Z).$$

$$\text{At } B_1 \text{ and } B_2 = \underline{+565 \text{ lb. } (Z).}$$

TABLE XIV.—SUMMARY OF LOADS IN MEMBERS.

Member.		Turning in Flight—Engine On: Gravity Loads.	Thrust.	Pitching: Left Turn	Pitching: Right Turn	Yawing	Torque	Total: Left Turn	Total: Right Turn.	Sideload	Inverted Flight	Landing.	Static Thrust and Torque
AE	Stbd	+1840	+ 54	+268	−268		+199	+2361	+1825	± 92	− 856	+ 856	Covered by other cases
	Port	+1840	+ 54	+268	−268		−199	+1963	+1427				
AB	*S*	−1353	−128	−236	+236	−550	−176	−2443	−1971	± 171	+ 756	− 756	
	P	−1353	−128	−236	+236	+550	+176	− 991	− 519				
BC	*S*	−1184	−205	−226	+226	−791	−192	−2598	−2146	± 516	+ 823	− 823	
	P	−1184	−205	−226	+226	+791	+192	− 632	− 180				
BE	*S*	+ 899	− 43	−145	+145	− 92	+116	+1025	+ 735	+ 20	− 501	+ 501	
	P	+ 899	− 43	−145	+145	+ 92	−116	+ 977	+ 687				
BF	*S*	.				−349		− 349	− 349	± 279	..	.	
	P		..	.		+349	.	+ 349	+ 349				
ED	*S*	+2760	+ 9	+127	−127	− 98	+317	+3115	+2861	± 112	−1362	+1362	
	P	+2760	+ 9	+127	−127	+ 98	−317	+2677	+2423				
EC	*S*	− 882	+ 43	+135	−135	+ 98	−113	− 719	− 989	± 18	+ 485	− 485	
	P	− 882	+ 43	+135	−135	− 98	+113	− 689	− 959				

TABLE XV.—ENGINE MOUNTING—STRENGTH OF MEMBERS.

Member.	End Load (lb.).	Case.	Length (in.).	Size.	Specification.	k.	A.	$\frac{l}{k}$.	Actual Stress lb./in.². Tension.	Actual Stress lb./in.². Compression.	Allowable Stress lb./in.². Tension.	Allowable Stress lb./in.². Compression	Reserve Factor (R.F.). Tension.	Reserve Factor (R.F.). Compression.	Remarks.
AB	2443 (T.) 756 (C.)	Engine on. Inv. Flight	23	¾-in. O/D × 20G.	T.45	·2528	·0807	91	30,200	9,380	101,000	30,000	3·35	3·2	
BC	2598 (T.) 823 (C.)	,,	19·7	,,	,,	,,	,,	78	32,200	10,200	,,	38,600	3 14	3·78	
AE	2361 (C.) 856 (T.)	,,	25 7	1-in. O/D × 20G.	,,	·3410	·1090	76	7,850	21,700	,,	40,500	> 5	1·87	ABC and AED are actually continuous members.
ED	3115 (C.) 1362 (T.)	,,	13 8	,,	,,	,,	,,	41	12,500	28,600	,,	71,000	> 5	2·48	
BE	1025 (C.) 501 (T.)	,,	12.0	,,	,,	,,	,,	35	18,000	32,300	,,	93,000	> 5	2·89	
EC	989 (T.) 485 (C.)	,,	24	,,	,,	,,	,,	71	9,080	4,450	,,	44,500	> 5	> 5	
BF	349 (C.) 349 (T.)	Engine on.	21 2	1-in. O/D × 22G.	,,	·3437	·0855	62	4,080	4,075	,,	53,000	> 5	> 5	

The method is that used in the worked example in "Struts," discussed earlier. The tube sizes given above are not necessarily the best that could be chosen, but will serve to illustrate the method of tabulation.

CHAPTER IX.

DIFFERENTIAL BENDING OF TWO-SPAR STRESSED-SKIN WINGS.

CONSIDER the loads perpendicular to the chord line at any section of a two-spar wing, the lift load L acting at the centre of pressure (C.P.) and the wing weight W at the centre of gravity (C G.) (see Fig. 64).

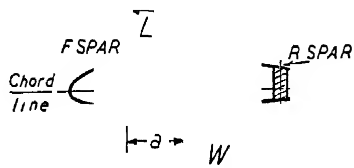


FIG 64.—Loading of a two-spar wing.

It is convenient to consider that the wing is twisting about some flexural centre F, so chosen that its distance a behind the centre line of the front spar is approximately

$$\frac{I_R}{I_F+I_R}\cdot d,$$

where

I_F = moment of inertia of front (F.) spar about a line through the section approximately parallel to the chord line,
I_R = *ditto* for rear (R) spar, and
d = spar centre distance.

The distance a will not, in general, be constant throughout the span, due to the variation in I_F, I_R and possibly d

We can consider L and W replaced by a net *direct* shear at the flexural centre equal to $(L-W)$, which will be divided between the F and R. spars in inverse proportion to the distance of the flexural line from them, combined with a torque T about the flexural centre equal to $(Lb+Wc)$. The values of b and c will usually vary throughout the span, due to the variation in the position of F, mentioned above.

The torque T can be plotted at various stations, part of it being taken by the spars in differential bending (as an upward shear on one spar and an equal downward shear on the other, the values being added algebraically to

those of direct shear found above), the remainder being taken by the skin The proportion taken by the spars is $Te^{-\mu x}$ and by the skin $T(1-e^{-\mu x})$, x being the distance along the wing from the root (usually) and

$$\mu=\sqrt{\frac{K}{d^2E}\left(\frac{1}{I_F}+\frac{1}{I_I}\right)}$$

where

$K = G \times$ Torsional Moment of Inertia (I_P),

$= G \times \dfrac{4A_1{}^2 t}{P}$;

d = distance between spar centres;

A_1 = area of torsion box enclosed by skin covering between forward face of F. spar and aft face of R. spar;

P = perimeter of section;

E = Young's Modulus of spar,

t = mean thickness of skin;

and G = the Modulus of Torsional Rigidity,

$= 1{\cdot}5 \times 10^5$ for ply covering.

Clearly, when $x=0$, $e^{-\mu x}=1$, so that at this point all the torque is taken by the spars in differential bending, but as x increases (i e. as we move outward from the root), more and more torque is taken by the skin.

The stressing of a two-spar wing will now be explained by a worked example.

Example 22.—Stressing of Two-Spar Wing.

Centre of Pressure Forward (C.P.F.) case. Load Factor (L.F.) = 9 0.
In Fig. 65 the all-up weight = 2000 lb.

$$\lambda=\frac{\text{chord at wing tip}}{\text{chord at wing root}}=\frac{C_T}{C_0}=\frac{53{\cdot}2}{77{\cdot}25}=0{\cdot}689.$$

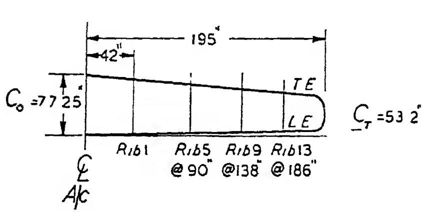


FIG. 65.—Diagram for Example 22.

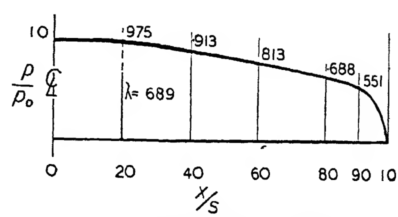


FIG. 66 —Load distribution diagram.

The load distribution diagram can be interpolated for this value of λ from Air Publication 970, VII, Fig. 3.

In Fig. 66, p = loading at any station, p_0 = loading at the centre line (C L.), unit loading at any station $= p/p_0$.

Integrate the loading curve to obtain the shear. Then, if the shear at the C.L $= F_0$ and the shear at any station $= F$, unit shear $= F/F_0$. The bending moment curve, also expressed as unit B.M. $= M/M_0$, is the integraph of the shear curve (see Fig. 67).

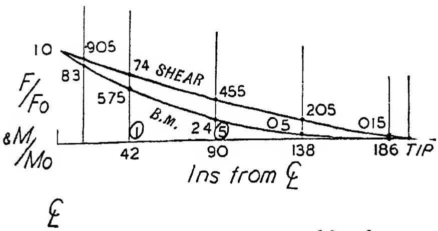


FIG. 67 —Unit air load shear and bending moment curves

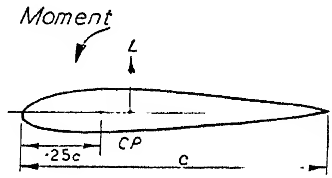


FIG 68 —Wing lift acting at the centre of pressure

Torque loading about quarter-chord point

The wing lift L, acting at the centre of pressure (C P.) of the wing (Fig 68), can be considered as a shear at quarter-chord, plus a moment, the value of the moment being $-\frac{1}{2}\rho SV^2c \times C_{MO}$ (lb. ft.) (the minus sign indicates a nosing-down moment), where

S = wing area (ft.²),
ρ = density of air = 00238 slugs/c. ft.,
c = chord (ft),
V = velocity (ft./sec.),

and C_{MO} = the moment coefficient which for this aerofoil section is 0·044

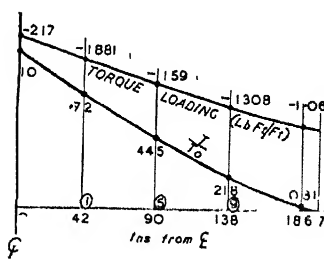


FIG 69 —Torque loading and unit torque about quarter-chord

$$\therefore \text{Moment} = -\tfrac{1}{2}\rho V^2c^2 \times C_{MO} \text{ lb. ft./ft.,} \quad \text{since } S = c \times 1 \text{ per ft. run}$$

$$- 0523c^2\frac{V^2}{10^3},$$

or

$$\text{Torque loading} \times \frac{10^3}{V^2} = \underline{-\cdot 0523c^2 \text{ lb. ft./ft.}}$$

This expression enables the torque loading about quarer-chord at each station to be found. by integration, the torque (lb ft.), expressed as unit torque (T/T_0), is obtained (see Table XVI and Fig. ~~69~~).
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TABLE XVI.—WING LOADS—TORQUE LOADING AND UNIT TORQUE.

Station.	Chord (in.)	Chord (c) (ft.)	c^2	Torque Loading $\times \frac{10^3}{V^2}$ $= -\cdot0523c^2$ lb. ft./ft.	Torque $\times \frac{10^3}{V^2}$ (by Integration) (lb. ft.)	Unit Torque $\frac{T}{T_0}$.
Rib 13	54	4·5	20·3	−1·061	− 0·78	0·031
,, 9	60	5·0	25	−1·308	− 5·51	0·218
,, 5	66	5·5	30·4	−1·590	−11·29	0·445
,, 1	72	6·0	36	−1·881	−18·23	0·72
C.L. m/c	77·25	6·44	41·5	−2·17	$T_0 = -25{\cdot}32$	1·00

$$\text{Wing moment} \times \frac{10^3}{V^2} = -25{\cdot}32$$

$$\text{Taking } V = V_S\sqrt{\frac{N}{2}} \quad 174 \text{ ft./sec.,}$$

$$\text{Wing moment } M = \frac{-25{\cdot}32 \times 174^2}{10^3} = -766 \text{ lb. ft./side (unfactored)}$$

$$= -1532 \text{ lb. ft./side (factored)}$$

(taking an additional factor of 2·0 as per Air Publication 970, II, 2).

Curves of $e^{-\mu x}$ and area of the torsion box A_1 are shown in Fig. 70,

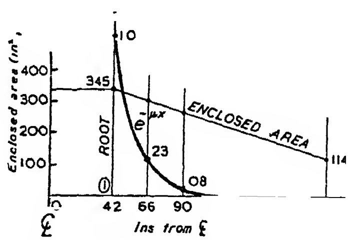

FIG. 70.—Curves of $e^{-\mu x}$ and area A_1.

but the detailed calculation will not be given, since it is the general method of attacking the problem, more than the intimate details, with which we are concerned.

Distribution of Wing-Weight Relief.

	Wt. per side
Outer Wing (from Rib 1 to Tip) . .	135 lb.
Centre-section Wing (from C.L. to Rib 1)	58 lb.
Fuel Tank and Fuel	121 lb.
	314 lb.

Assume that over the outer wing the weight is proportional to the product of the chord × maximum thickness, and that over the centre-section (c/s) wing the weight is uniformly distributed.

The weight of the tank and fuel will be taken off as concentrated shears = 60·5 lb. each at stations 25 in. and 31 in. from the centre line (see Fig. 71).

Table XVII relates to the outer wing.

TABLE XVII.—OUTER WING—WEIGHT RELIEF SHEAR.

Station.	Chord (in.).	Thickness (in.).	Product (in.^{2}).	Weight (lb.).	Shear (lb.) by Integration.
Rib 13	54	4.9	265	17.5	4
,, 9	60	6.9	414	27.4	33
,, 5	66	8.80	584	38.6	76
,, 1	72	10.8	778	51.5	135

Total = 135 lb.

Centre-Section Wing.—The shear diagram will be triangular, with ordinate = 58 lb. at the centre line (C.L.), i.e. the shear at the C.L. due to wing weight = 58 + 135 = 193 lb., the total shear at the C.L., including tank and fuel, being 314 lb.

The unit weight relief shears, i.e. expressed as fractions of the shear at the C.L. (= 314 lb.), are marked on the shear diagram, Fig. 71.

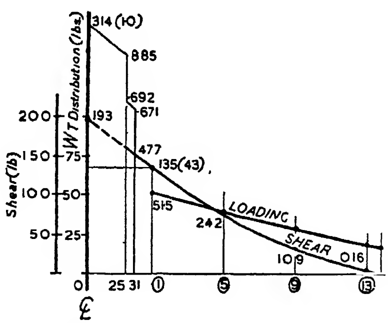

FIG. 71.—Wing-weight relief shear per side (unfactored).

Assuming that we know from balance calculations that—

Total lift L per side = 8830 lb.,
Total drag D per side = 1212 lb.,
Inertia force F = ·691 lb./lb., and
Attitude (α) of wing in C.P.F. case = 18 deg.,

we can now resolve these loads normal to and along the chord line.

(*a*) *Lift and Drag.*

Resultant normal to chord line

$$=L \cos \alpha + D \sin \alpha$$
$$=8830 \times \cdot 9511 + 1212 \times \cdot 309$$
$$=8400 + 375 = \underline{8775 \text{ lb}} \text{ (upward).}$$

Resultant along chord line

$$=L \sin \alpha - D \cos \alpha$$
$$=2730 - 1152 = \underline{1578 \text{ lb.}} \text{ (forward).}$$

(*b*) *Gravity Relief and Inertia Forces.*

$$W = \text{Wing-weight relief}$$
$$= -9 \times 314 = \underline{-2826 \text{ lb}}$$
$$F = \text{inertia force} = \cdot 691 \times 628 = \underline{434 \text{ lb}}$$

Resultant normal to chord line

$$= -W \cos \alpha - F \sin \alpha$$
$$= -2680 - 134$$
$$= \underline{-2814 \text{ lb}} \text{ (downward)}$$

Resultant along chord line

$$= -W \sin \alpha + F \cos \alpha$$
$$= -872 + 413$$
$$= \underline{-459 \text{ lb}}$$

Forces Normal to Chord—Direct Shear in Spars.—We have just found that the resultant air load normal to the chord line at the C.L. of the machine is 8775 lb., so by referring to our unit air load shear diagram (Fig. 67) we can obtain the shear at various stations by multiplying this value of 8775 by the appropriate factor (col. 2) in Table XVIII

TABLE XVIII.—SHEAR IN SPARS

(1) Station	(2) Factor.	(3) Air Load Shear (N) (lb).	(4) Factor.	(5) Relief Shear (R) (lb)	(6) Net Shear at Flexural Centre (lb).	(7) Factor.	(8) Front Spar Shear (lb).	(9) Rear Spar Shear (lb)
C L. m/c.	1 0	8775	1 0	−2814	+5961	67	+4000	+1961
25 in.	·84	7360	885	−2490	+4870	·67	3260	1610
			692	−1950	+5410		3610	1800
31 in.	805	7060	671	−1885	+5175	·67	3460	1715
			477	−1340	+5720		3820	1900
Rib 1	74	6500	43	−1210	+5290	67	3540	1750
,, 5	455	4000	242	− 682	+3318	658	2180	1138
,, 9	205	1800	109	− 307	+1493	642	960	533
,, 13	015	132	016	− 45	+ 87	628	55	32

Similarly, the relief shear throughout the span (col. 5) can be obtained by factoring the weight relief shear (=2814 lb) by the values in col. 4.

Shear in Spars (+upward).—The factor in col. 7 is the proportion of the direct shear taken by the front spar; it is found by plotting the position of the flexural line relative to the spars throughout the span (this curve is not shown here).

The values in cols. 6, 8 and 9 are then plotted as in Fig. 72.

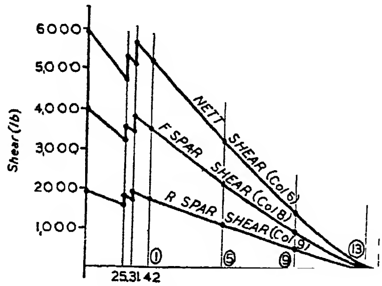

FIG. 72 —Resultant direct shear normal to chord

Forces Along Chord.—By a similar procedure the resultant shear along the chord is obtained (see Table XIX). (Curves are not drawn)

TABLE XIX.—SHEAR ALONG THE CHORD (+FORWARD).

Station.	Air Load Shear along Chord.	Relief Shear along Chord	Net Shear along Chord
C.L m/c	1578	-459	+1119
25 in	1320	-407 -319	+ 913 +1001
31 in.	1270	-308 -219	+ 962 +1051
Rib 1	1170	-198	+ 972
„ 5 .	717	-111	+ 606
„ 9	323	- 50	+ 273
„ 13 .	24	- 7	+ 17

Torque.—Referring to Fig. 73, it will be seen that the *resultant torque about the flexural centre F* =moment of air load shear (N), plus the moment of the relief shear (R), minus the moment (M) about quarter-chord.

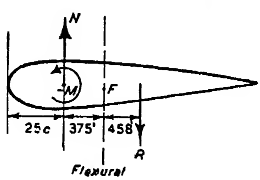

FIG. 73.—Diagram for Table XX.

Values for the resultant torque T at various stations are given in Table XX. From these values we may find the torque in the spars $(Te^{-\mu x})$ and that in the skin $\{T(1-e^{-\mu x})\}$ (see Table XXI and Fig. 74).

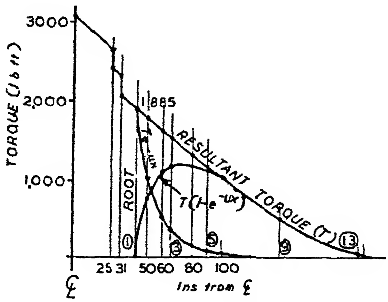

FIG. 74.—Torque curves.

TABLE XX.—TORQUE AT VARIOUS STATIONS (+NOSING UP).

Station	Moment of Air Load Shear (N) at Quarter-chord about Flexural Line M_N (lb. ft.).	Moment of Relief Shear (R) about Flexural Line M_R (lb. ft.).	Moment about Quarter-chord M (lb. ft.).	Resultant Torque T (lb ft.).
C.L. m/c. .	·375 × 8775 = +3290	·458 × 2814 = +1290	−1532	+ 3048
25 in. .	375 × 7360 = +2760	·458 {2490 = +1142 ·458 {1950 = + 894	−1265	+ 2637 + 2389
31 in. .	375 × 7060 = +2650	458 {1885 = + 865 458 {1340 = + 615	−1210	2305 2055
Rib 1 .	375 × 6500 = +2430	458 × 1210 = + 555	−1100	1885
,, 5 .	·375 × 4000 = +1500	458 × 682 = + 313	− 681	1132
,, 9 .	·375 × 1800 = + 675	·458 × 307 = + 141	− 337	479
,, 13 .	375 × 132 = + 50	·458 × 45 = + 21	− 46	25

TABLE XXI.—TORQUE IN SPARS AND SKIN.

Station.	T (lb. ft.)	$e^{-\mu x}$.	$Te^{-\mu x}$ (lb. ft.).	$T(1-e^{-\mu x})$ (lb ft.).
Rib 1 . .	1885	1·0	1885	..
50 in. . .	1750	·57	998	752
60 in. . .	1590	·325	516	1074
Rib 3 (66 in.) .	1500	·23	345	1155
80 in. . .	1290	·095	123	1167
Rib 5 (90 in.) .	1130	·04	45	1085
100 in. . .	1000	·01	10	990

The shear in the spars due to torque is then equal to

$$\frac{\text{Torque in spars}}{\text{Distance between spar centres}},$$

thus giving the total shear in the front and rear spars (Tables XXII and XXIII).

By integration, the bending moment in each spar can then be obtained. The tabulation is self-explanatory.

TABLE XXII.—SHEAR IN SPARS DUE TO TORQUE.

Station.	Torque in Spars (lb. ft.) $Te - \mu x$	Distance between Spar Centres (ft.)	Shear in Spars (lb.) (adds to Front Spar, subtracts from Rear Spar).
C.L. m/c	3048	2.96	1030
25 in. Inboard	2637	2.96	890
25 in. Outboard	2389		806
31 in. Inboard	2305	2.96	778
31 in. Outboard	2055		694
Rib 1 (42 in.)	1885	2·96	636
,, 2 (54 in.)	780	2·898	270
,, 3 (66 in.)	345	2.835	122
,, 4 (78 in.)	150	2.773	54
,, 5 (90 in.)	45	2.71	17

TABLE XXIII.—TOTAL SHEAR IN SPARS.

	FRONT SPAR			REAR SPAR		
Station.	Bending Shear (lb.)	Torque Shear (lb.).	Net Shear (lb.).	Bending Shear (lb.).	Torque Shear (lb.).	Net Shear (lb.).
C.L. m/c	+4000	+1030	+5030	+1961	−1030	+ 931
25 in. Inboard	+3260	+ 890	+4150	+1610	− 890	+ 720
25 in. Outboard	+3610	+ 806	+4416	+1800	− 806	+ 994
31 in. Inboard	+3460	+ 778	+4238	+1715	− 778	+ 937
31 in. Outboard	+3820	+ 694	+4514	+1900	− 694	+1206
Rib 1	+3540	+ 636	+4176	+1750	− 636	+1114
,, 2	+3180	+ 270	+3450	+1600	− 270	+1330
,, 3	+2850	+ 122	+2972	+1450	− 122	+1328
,, 4	+2500	+ 54	+2554	+1300	− 54	+1246
,, 5	+2180	+ 17	+2197	+1138	− 17	+1121
,, 9	+ 960	..	+ 960	+ 533	..	+ 533
,, 13	+ 55	..	+ 55	+ 32	..	+ 32

PART II.

DETAIL STRESSING.

STRESSING may be divided into two broad classes, the first of which, Primary Stressing, is concerned with finding the loads on a structure as a result of certain conditions, aerodynamic or otherwise, usually laid down in Air Ministry requirements, whilst the other class, Detail Stressing, concerns the strength calculations of individual members, joints, fittings, etc., when these loads are applied.

In the drawing office, the draughtsman is usually given certain loads and, after working out a preliminary design scheme, desires to make a strength check. The method of doing this is illustrated below by means of a varied assortment of worked examples, taken from actual practice.

It should be noted, however, that the *complete* stressing of each example has not been attempted; this would mean a considerable amount of overlapping, whereas the aim throughout has been to bring out some fresh point in each case.

Emphasis must be laid on the fact that detail stressing is not an exact science: the methods given in the succeeding pages can at the most, therefore, serve only as a guide to assumptions that are considered reasonable in any particular case, the results obtained being, in the main, pessimistic, and consequently erring on the right side.

Example 23.—Plate Fitting.

Consider a duralumin plate (specification L.3, 10 G. ($=0{\cdot}128$ in.) thick), acted on by a factored load of 600 lb. tension, as shown in Fig. 75.

Bursting Strength of Plate.

The load will tend to shear out the portion *abcd* of the plate, the area resisting shear (bursting) being $2ab \times t$, where t is the thickness of the plate. (*Note.*—All dimensions are in inches.)

It is customary in practice, however, to use, instead of $2ab$, an arbitrary value of $1{\cdot}75e$, where e is the distance from the edge of the hole to the edge of the plate, the area resisting bursting thus being $1{\cdot}75\, e \times t$ (in.^{2}).

If f_s = the allowable shear stress of the material (lb./in.²), the allowable load (lb.) $= 1{\cdot}75\ e \times t \times f_s$.

In this example,

$f_s = 30{,}000$ lb./in.² for L.3,

$e = {\cdot}35 - {\cdot}125 = {\cdot}225$ in. and

$t = {\cdot}128$ in.;

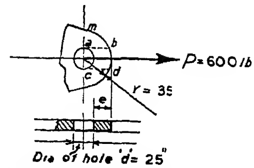

FIG 75.—Plate acted on by a factored load of 600 lb. tension.

i.e. the allowable bursting load

$$= 1{\cdot}75 \times {\cdot}225 \times {\cdot}128 \times 30{,}000 = 1500 \text{ lb.}$$

and the Reserve Factor (R.F.) $= \dfrac{\text{Allowable load}}{\text{Actual factored load}} = \dfrac{1500}{600} = \underline{2{\cdot}5}.$

Sometimes, in order to increase the bursting strength whilst still keeping the same gauge of plate, the radius r of the end is not struck from the same centre as the bolt hole. For example, suppose r were struck from the edge of the $\frac{1}{4}$-in. hole (Fig. 76), e would now be ·35 in, and

$$\text{R F} = \frac{{\cdot}35}{{\cdot}225} \times 2{\cdot}5 = \underline{3{\cdot}88}.$$

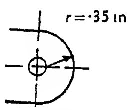

FIG 76 —Method of increasing the bursting strength, while still keeping the same gauge of plate.

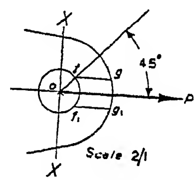

FIG 77.—Alternative method for finding the bursting strength.

In cases where the design is critical, that is, when the first method gives a R.F. just below 1·0 and it is not possible to increase r or t, the following more accurate calculation can be made.

From the centre of the bolt hole draw two radial lines (Fig. 77) of and of_1 at 45° to the axis of the load, and where they cut the bolt circle draw fg and f_1g_1 parallel to the axis. Measure fg. (For this purpose it is usually better to draw a "twice-full-size" view.)

Then the area resisting shear is $1{\cdot}75 fg \times t \times f_s$.

In this example, $fg = {\cdot}25$ in.

$$\text{and} \quad \text{R F.} = \frac{{\cdot}25}{{\cdot}225} \times 2{\cdot}5 = 2{\cdot}78.$$

Bearing Strength of Plate.

When considering the strength of the plate to resist bearing (or crushing, as it is sometimes called) by the bolt, the bearing area is taken as $d \times t$.

Thus, if f_B is the allowable bearing stress in lb./in.^{2} ($=70{,}000$ lb /in.^{2} for L.3),

RESERVE FACTOR

$$\text{Allowable bearing load} = d \times t \times f_B,$$
$$= \cdot 25 \times \cdot 128 \times 70{,}000,$$
$$= 2240 \text{ lb.}$$
$$\text{Actual load} = 600 \text{ lb.}$$ 3·73

Strength of Plate in Tension at XX.

Net area of cross-section at $XX = t \times 2am$ (see Fig. 75).

$$\text{Taking} \quad am = \cdot 23 \text{ in.},$$
$$\text{area} = \cdot 128 \times \cdot 46$$
$$= \cdot 0588 \text{ in.}^2.$$

Allowable tension $= \cdot 0588 \times f_t$, where f_t = allowable tensile stress (45,000 lb./in.^{2} for L.3)

$$= 2600 \text{ lb.}$$
$$\text{Actual Load} = 600 \text{ lb.}$$ 4·34

Assume that the fitting under discussion is of the type shown in Fig. 78.

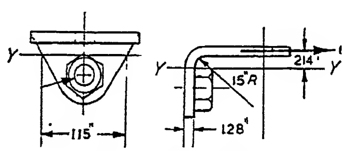

FIG. 78.—Detail of fitting.

There will be bending in the plate at section YY just outside the bolt head.

$$\text{B.M.} = 600 \times \cdot 214 = 128 \text{ lb. in.}$$

$$Z = \frac{bd^2}{6} = \frac{1\cdot 15 \times \cdot 128^2}{6} = \cdot 00315 \text{ in.}^3$$

$$\text{Bending stress} = \frac{128}{\cdot 00315} = 40{,}700 \text{ lb./in.}^2$$

Allowable bending stress for L.3 $= 45{,}000$ lb /in.^{2} 1·10

Tension in $\frac{1}{4}$-in. bolt $= 600$ lb.

Allowable tensile load in $\frac{1}{4}$-in. mild steel bolt (specification S.1) $= 2460$ lb. 4·1

Example 24.—Bracket Machined from Duralumin Bar (Specification L.1).

Applied Load = 2000 lb. as shown in Fig. 79. RESERVE FACTOR

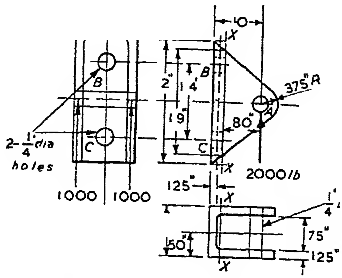

FIG. 79.—Detail of bracket

Take the allowable bearing stress for L.1 = 70,000 lb./in.^{2}

" " bending " " = 39,000 lb./in.^{2}

" " shear " " = 30,000 lb./in.^{2}

Bearing strength of bolt in bracket at A

$$= \cdot 25 \times \cdot 125 \times 70{,}000 = 2190 \text{ lb.}$$

Load per side = 1000 lb. 2·19

Shear strength of $\frac{1}{4}$-in. bolt at A—

taking the allowable double shear strength = 4380 lb. 2·19

Tension in bolt C due to offset

$$= \frac{2000 \times (1\cdot 0 - \cdot 125)}{1\cdot 4} = 1250 \text{ lb.}$$

Allowable tension in $\frac{1}{4}$-in. bolt = 2460 lb. 1·97

Shear in bolts B and C = 1000 lb. each.

Single shear strength = 2500 lb. 2·5

Bearing strength of bolt in bracket at C—

As at A. 2 19

Bending strength at XX:

$$Z = 2 \cdot \frac{bd^2}{6} = \frac{\cdot 125 \times 1\cdot 9^2}{3} = \cdot 15 \text{ in}^3$$

$$M = 2000 \times \cdot 80 = 1600 \text{ lb. in.}$$

Bending stress $= \dfrac{1600}{\cdot 15} =$ 10,680 lb./in.^{2} 3 65

Bursting strength:

At C—clearly up to strength.

At A—does not enter into the calculation with load as shown.

Example 25.—Plug-end : Duralumin Bar (Specification L.1).

Load : 745 lb. (as shown in Fig. 80). RESERVE FACTOR

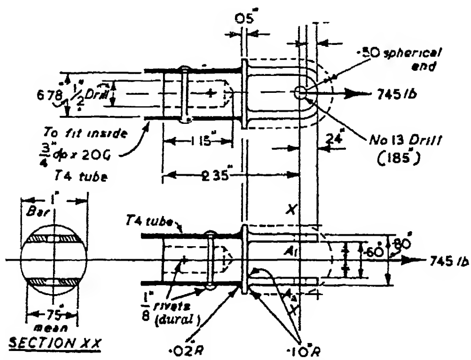

FIG. 80.—Detail of plug end.

Bearing strength at A_1 and A_2

$=2\times{\cdot}185\times{\cdot}10\times70{,}000=2590$ lb.: at 745 lb. **3·48**

Bursting strength at A_1 and A_2

$=2\times1{\cdot}75\times{\cdot}10\times{\cdot}12\times30{,}000=1260$ lb.: at 745 lb. **1·69**

Bearing strength of $\frac{1}{8}$-in. (dural.) rivets in dural. (T.4) tube (there are 4 bearing surfaces)—

$4\times{\cdot}125\times{\cdot}036\times70{,}000=1260$ lb.: at 745 lb. **1·69**

Taking shear strength of rivet = 356 lb.,

$$\text{Load per face}=\frac{745}{4}=186\text{ lb.}$$ **1·91**

Bearing strength of rivets in plug-end

$=4\times{\cdot}125\times{\cdot}089\times70{,}000=6240$ lb.: at 745 lb. **> 5**

Note.—It is usual to state any R.F. which is over 5 as greater than 5, or > 5.

Very often, too, it can be seen by inspection that the strength is satisfactory. For instance, the bursting strength at rivet-hole *B* in the plug-end is fairly obviously O.K. by inspection.

Strength in tension at *XX*:

Area $=2\times{\cdot}10\,({\cdot}75-{\cdot}185)={\cdot}113$ in.^{2}.

Tensile stress $=\dfrac{745}{{\cdot}113}=6600$ lb./in.^{2}: at 56,000 lb./in.^{2}. **> 5**

Example 26.—Universal Joint Bracket (S.1).

Torque is applied as shown in Fig. 81 $=1220$ lb. in. RESERVE FACTOR

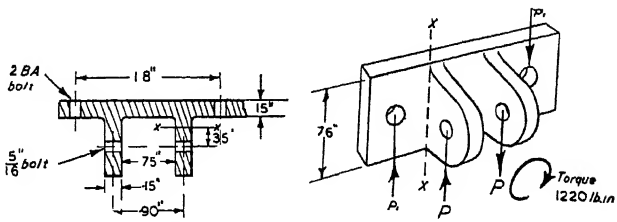

FIG 81.—Universal joint bracket

Shear (P_1) in 2 B.A. bolts due to torque $\dfrac{1220}{\ }=678$ lb.

Allowable single shear in 2 B.A. bolt $=1370$ lb.: 2·02

Bearing strength of 2 B A bolt in bracket

$$=\cdot 185\times\cdot 15\times 117{,}000=3240 \text{ lb : at } 678 \text{ lb.}$$ 4·77

Load P in $\frac{5}{16}$-in. bolt $=\dfrac{1220}{\cdot 90}$ 1360 lb.

Bearing strength of bracket at $\frac{5}{16}$-in. hole

$$=\cdot 3125\times\cdot 15\times 117{,}000 \text{ lb.}=5500 \text{ lb : at } 1360 \text{ lb.}$$ 4·05

Shear strength of $\frac{5}{16}$-in bolt $=3910$ lb.: at 1360 lb. 2·88

Bending strength at XX:

$$\text{B.M.}=1360\times\cdot 35=476 \text{ lb. in.}$$

$$Z=\frac{\cdot 15\times\cdot 76^2}{6}=\cdot 0145 \text{ in.}^3$$

$$\text{Bending stress}=\frac{476}{\cdot 0145}=32{,}800 \text{ lb./in.}^2\text{: at } 69{,}000 \text{ lb /in.}^2.$$ 2·1

Example 27.—Lever: Specification, Nickel Chrome Steel Bar (S.11).

Load: 50 lb. applied as shown, giving combined Torsion and Bending at section XX (see Fig. 82)

$$\text{Torque } T=50\times 1\ 6=80 \text{ lb. in.}$$
$$\text{B.M. at } XX=50\times\cdot 40=20 \text{ lb. in}$$

RESERVE FACTOR

For solid rectangle—

Shear stress $q=\frac{T}{ab^2}\left(3+1{\cdot}8\frac{b}{a}\right)$ $\qquad b=\cdot104;\quad \frac{b}{a}=\cdot186;$

$a=\cdot56;$

$b^2=\cdot0108.$

$$q=\frac{80}{\cdot56\times\cdot0108}(3+1{\cdot}8\times\cdot186)$$

$$=13{,}200\ (3+\cdot334)$$

$$=43{,}900\ \text{lb./in.}^2$$

$$Z=\frac{\cdot104\times\cdot56^2}{6}=\cdot00541\ \text{in.}^3.$$

Bending stress $p=\frac{20}{\cdot00541}=3690$ lb./in.^{2}.

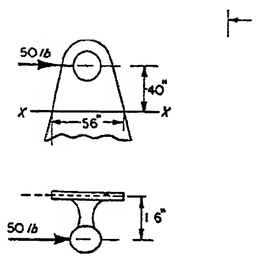

FIG. 82.—Load of 50 lb. applied to lever as shown, giving combined torsion and bending at XX.

(a) Principal direct stress (see Air Publication 970, VIII, (III), 3)

$$=\frac{p}{2}+\sqrt{\left(\frac{p}{2}\right)^2+q^2}$$

$$=1845+\sqrt{1845^2+43{,}900^2}$$

$$=1845+10^3\sqrt{3{\cdot}41+1935}$$

$$=1845+44{,}000$$

$=45{,}845$ lb./in.^{2}: at 110,000 lb./in.^{2} 2·4

(b) Maximum shear stress

$$=\sqrt{\left(\frac{p}{2}\right)^2+q^2}$$

$=44{,}000$ lb./in.^{2} from above: at 76,000 lb./in.^{2} 1·72

Example 28.—Bracket.

Loads applied: 2180 lb. as shown, giving resultant downward shear of 1440 lb. on bracket (see Fig. 83).

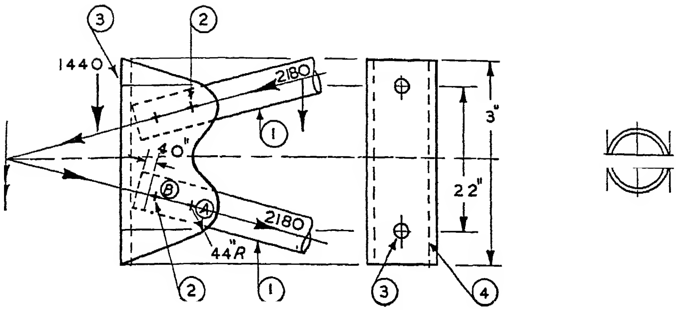

FIG 83 —Detail of bracket

Schedule :

Ref.		Spec.
(1)	$\frac{3}{4}$-in O/D × 20G. Tube	T.45
(2)	$\frac{3}{16}$-in. O/D × 20G Tubular Rivets	T.35
(3)	$\frac{1}{4}$-in Bolt	A 1
(4)	20G Mild Steel Plate.	S 3

Resultant downward shear on each $\frac{1}{4}$-in. bolt (3) $= \frac{1440}{2}$ lb. $= 720$ lb. RESERVE FACTOR

Allowable shear on $\frac{1}{4}$-in. A.1 $= 2750$ lb. 3·82

Bearing strength of $\frac{1}{4}$-in. bolts (3) in bracket (4)

$= \cdot 25 \times \cdot 036 \times 94{,}000 = 846$ lb.: at 720 lb. 1·17

$\frac{3}{16}$-in. O/D × 20G. tubular rivets (2):

Take shear strength $= 740$ lb.

Load per rivet per face $= \frac{2180}{4} = 545$ lb. 1·36

Bearing strength in bracket (4)

$= \cdot 1875 \times \cdot 036 \times 94{,}000 = 634$ lb.: at 545 lb. 1·16

To increase the bearing area, fit $\frac{1}{4}$-in. O/D × 22G. T 35 distance tubes. (This is the customary procedure, but an amendment would have to be made in the Schedule to this effect.)

Bearing strength $= \cdot 25 \times \cdot 036 \times 94{,}000 = 846$ lb : at 545 lb. 1·55

Bursting strength of bracket at *A* when distance tubes are fitted

$= 1 \cdot 75 \times (\cdot 44 - \cdot 125) \times \cdot 036 \times 45{,}000 = 895$ lb.: at 545 lb. 1·64

RESERVE FACTOR

Bursting strength of tube at B

$=1{\cdot}75\times({\cdot}40-{\cdot}125)\times{\cdot}036\times59{,}000=1020$ lb.: at 545 lb.

Bearing strength of bracket at A

$={\cdot}25\times{\cdot}036\times94{,}000=846$ lb. : at 545 lb. 1·55

Bearing strength of tube at A

$={\cdot}25\times{\cdot}036\times151{,}000=1360$ lb.: at 545 lb. 2·5

Example 29.—Lever and Spool Assembly.

Loads: as shown in Fig. 84.

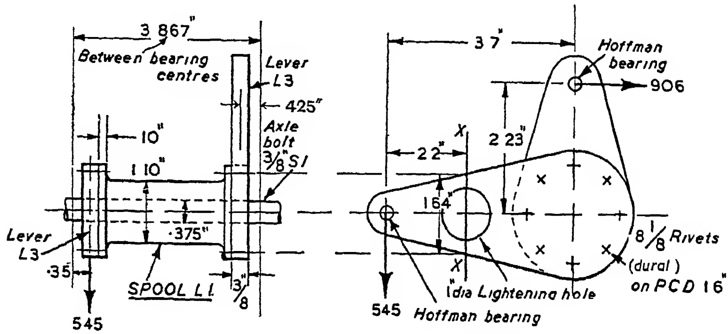

FIG. 84.—Detail of lever and spool assembly.

Bending strength at section XX at lightening hole:

$$M=545\times2{\cdot}2=1200 \text{ lb. in.}$$

$$I=\frac{{\cdot}375}{12}[1{\cdot}64^3-1{\cdot}0^3]={\cdot}107 \text{ in.}^4.$$

Bending stress $=\dfrac{1200\times{\cdot}82}{{\cdot}107}=9200$ lb./in.^{2}. L.3 at 45,000 lb./in.^{2}. >5

Rivets.—Torque applied to rivets $=545\times3{\cdot}7=2020$ lb. in.

Torque shear in each rivet $=\dfrac{1}{8}\times2020\times\dfrac{2}{1{\cdot}6}=315$ lb., since the pitch circle diameter (P.C.D.) $=1{\cdot}6$ in.

Direct shear for rivet $=\dfrac{545}{8}=68$ lb.

Net ,, ,, ,, $=383$ lb.

Allowable shear in $\frac{1}{8}$-in. duralumin rivet $=356$ lb. 0·93

The rivets are therefore down in strength.

Bearing strength of rivets in spool—

${\cdot}125\times{\cdot}10\times70{,}000=875$ lb.: at 383 lb. 2·28

RESERVE FACTOR

Bearing strength of rivets in lever—

$\cdot 125 \times 375 \times 70{,}000 = 2625$ (clearly covered at 383 lb) > 5

Allowable load in Hoffmann bearing = 1050 lb at 906 lb. 1·16

Axle Bolt.—$\frac{3}{8}$-in. S 1 (will carry bending and shear but not torque, which will be carried by spool)

Loads acting are:

545 lb downward at one end

906 lb. at right angles at other end (see Fig 85).

In plane of 545-lb. load:

Reactions:

$$R_2 = \frac{545 \times 3{\cdot}517}{3{\cdot}867} = 496 \text{ lb.}$$

$$R_1 = 49 \text{ lb.}$$

Bending moment $M = 496 \times \cdot 35 = 174$ lb in.

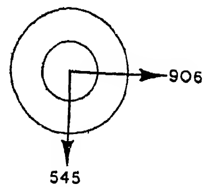

FIG. 85.
Loads on axle bolt.

In plane of 906-lb. load.

$$R_1 = \frac{906 \times 3\ 442}{3\ 867} = 805 \text{ lb.,} \qquad R_2 = 100 \text{ lb.}$$

$$M = 805 \times 425 = 342 \text{ lb. in}$$

Bending moment under 906 lb load due to 545 lb. load

$$= 49 \times \cdot 425 = 21 \text{ lb. in.}$$

$$\text{Net } M = \sqrt{21^2 + 342^2} = 343 \text{ lb in. (say)}$$

Bending stress:

$$Z = \frac{\pi D^3}{32} = \frac{\pi \times \cdot 375^3}{32} = \cdot 00518 \text{ in.}^3$$

$$f = \frac{343}{\cdot 00518} = 66{,}200 \text{ lb/in.}^2$$ S.1 at 69,000 lb./in. 1·04

Maximum shear in axle bolt will be at the supports, and will be the maximum resultant reaction there

From above, maximum shear at (2) $= \sqrt{496^2 + 100^2} = 506$ lb.

and at (1) $= \sqrt{805^2 + 49^2} = 807$ lb.

Shear strength of $\frac{3}{8}$-in. S.1 = 5630 lb > 5

RESERVE FACTOR

Spool.—Torque in spool $=906\times 2{\cdot}23=2020$ lb in

Torque shear stress

$$\frac{16TD}{\pi(D^4-d^4)} \qquad \begin{aligned} D^4&=1{\cdot}10^4=1{\cdot}470\\ d^4&={\cdot}375^4=\ {\cdot}019\\ &\qquad 1{\cdot}451\end{aligned}$$

$$=\frac{16\times 2020\times 1{\cdot}1}{\pi\times 1{\cdot}451}=7800 \text{ lb./in.}^2\text{: at } 30{,}000 \text{ lb./in.}^2 \qquad 3{\cdot}85$$

Example 30.—Engine-Mounting Joint. (Fig. 86.)

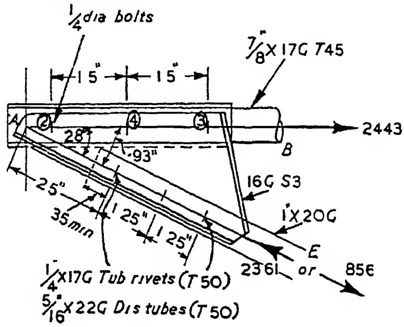

FIG. 86.—Details of engine-mounting joint.

Loads in lb.: in $AB=2443$ (T.).
in $AE=2361$ (C.) or 856 (T.).

Bolts (2), (3) and (4) will have to carry loads in plate from AE only.

Rivets in AE:

Allowable shear load in $\frac{1}{4}$-in. $\times$ 17G. T.50 rivet $=1350$ lb.

$$\text{Load/face}=\frac{2361}{6}=394 \text{ lb.} \qquad 3{\cdot}42$$

Bearing strength in 16G. plate (S.3)—

$${\cdot}25\times{\cdot}064\times 94{,}000=1500 \text{ lb.: at } 394 \text{ lb.} \qquad >5$$

Bursting strength at (1) under 856 lb. (T.)—

$$1{\cdot}75\times{\cdot}036\times{\cdot}20\times 59{,}000=743 \text{ lb.: at } \frac{856}{6}=143 \text{ lb.} \qquad >5$$

Bolts in AB:

Due to offset of 2361 lb. (assumed taken by (2) and (3))—

$$\text{Shear}=\frac{2361\times{\cdot}93}{3}=730 \text{ lb.,}$$

RESERVE FACTOR

Direct shear $= \dfrac{2361}{2} = 1180$ lb.;

Resultant shear (see Fig. 87) $= 1650$ lb. $= 825$ lb./face.

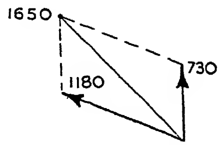

FIG. 87.—Vector diagram of direct and resultant shears.

Allowable shear for $\frac{1}{4}$-in. bolt $= 2500$ lb. 3·03

Bearing in 16G. S.3. Strength $= 1500$ lb. (see above) at 825 lb. 1·82

Bearing in $\frac{7}{8}$-in. $\times$ 17G tube (AB) T.45—

$\cdot 25 \times \cdot 056 \times 151{,}000 = 2110$ lb.: at 825 lb. 2·56

Stability of 16G. S.3 plate considered satisfactory by inspection.

Tension in AB·

Area of section ($\frac{7}{8}$-in. $\times$ 17G) $= \cdot 1441$ in.^{2}

2 $\frac{1}{4}$-in. holes $= \cdot 25 \times 2 \times \cdot 056 = \cdot 028$ in ^{2}

Net area $= \cdot 1161$ in ^{2}

Tensile stress $= \dfrac{2443}{\cdot 1161} = 21{,}000$ lb./in.^{2}: at 101,000 lb./in ^{2} 4·81

Example 31.—Shear Load on Welded Joint and Taper Pins. (Fig. 88.)

Applied torque $= 1340$ lb. in.

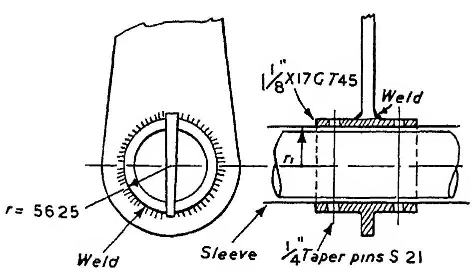

FIG. 88 —Detail of joint.

The shear load on a weld is usually stated as so much per inch run.

The torque of 1340 lb. in is resisted by the weld in shear, and gives a shear load RESERVE FACTOR

$$\frac{1340}{\cdot 5625}=2390 \text{ lb.}$$

and a shear load per inch run of weld

$$\frac{2390}{2\pi r}=676 \text{ lb.}$$

Allowable shear load per inch run for 17G T 45 = 1680 lb. 2·48

Shear load on taper pins:

$$r_1=\tfrac{1}{2}[1\cdot 125-2\times\cdot 056]=\cdot 506 \text{ in.}$$

Shear load per face of taper pins $=\dfrac{1340}{4\times\cdot 506}=662$ lb.

Allowable shear load on $\frac{1}{4}$-in. S.21 taper pin = 1770 lb. 2·67

Bearing strength of taper pin in 17G. T.45

$$=\cdot 25\times\cdot 056\times 151{,}000=2120 \text{ lb.: at } 662 \text{ lb.}$$ 3·2

The bearing strength in the sleeve would have to be checked too.

Eccentric Load on Bolted (or Riveted) Fitting.

The offset load can be replaced by a direct shear at the C.G. of the bolt group plus a torque about the C.G., this torque giving on each

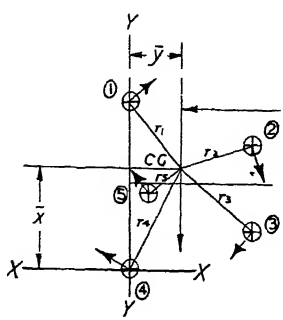

FIG. 89.—Eccentric load on bolted (or riveted) fitting.

bolt an additional shear acting at right angles to the line joining the C.G. and the bolt centre.

The resultant shear is found by compounding the direct and torque shears on each bolt, either graphically or otherwise.

Case I.—Bolts (Rivets) of Different Diameter.

If F = the applied load,
M = torque = $F \times a$,
r_1, r_2, etc are the distances from C.G. to bolt centres, and
D_1, D_2, etc. are the bolt diameters,

Direct shear on any bolt can be shown to be—

$$S_1 = F \times \frac{D_1}{\Sigma D}, \qquad S_2 = F \times \frac{D_2}{\Sigma D}, \quad \text{etc.};$$

and Torque shear on any bolt is—

$$P_1 = \frac{M \times D_1 r_1}{\Sigma D r^2}. \qquad = \frac{M \times D_2 r_2}{\Sigma D r^2}$$

The problem is best tackled by tabulation

First of all find the position of the C G. of the bolt group by taking moments about any convenient datums XX and YY (to do this find the area of each bolt, A_1, A_2, etc , and its distance x_1, x_2, etc. from XX and y_1, y_2, etc. from YY). Tabulate for x and y thus —

Bolt (or Rivet)	Dia. D (in.)	A (in ²)	x (in)	Ax (in ³)	$\bar{x} = \frac{\Sigma Ax}{\Sigma A}$ (in)	y (in).	Ay (in ³)	$\bar{y} = \frac{\Sigma Ay}{\Sigma A}$ (in)
(1)								
(2)								
(3)								
Etc								

ΣA = $\qquad$ ΣAx = $\qquad$ ΣAy =

For the offset shear set out thus:—

Bolt (or Rivet).	Dia D (in)	r (in)	r^2	Dr	Dr^2.	$\frac{Dr}{\Sigma Dr^2}$.	$P = \frac{MDr}{\Sigma Dr^2}$	Direct Shear $S = \frac{FD}{\Sigma D}$.	Resultant Shear R (graphically)
(1)									
(2)									
(3)									
Etc									

ΣD = $\qquad$ ΣDr^2 =

Case II.—Bolts (Rivets) all of Same Diameter

On the assumption that the offset shear on any bolt varies as r, i.e.

$$\frac{P_1}{P_2}=\frac{r_1}{r_2} \quad \text{or} \quad P_1r_2=P_2r_1, \quad \text{etc.}$$

Torque resisted by bolt (1) $=P_1r_1$.

$$\text{,,} \qquad \text{,,} \qquad \text{,,} \quad (2)=P_2r_2=\frac{P_1r_2}{r_1}r_2=\frac{P_1r_2^2}{r_1}.$$

$$(3)=P_3r_3 \qquad \frac{P_1r_3^2}{r_1}$$

$$\text{Total resistance } M=P_1r_1+\frac{P_1r_2^2}{r_1}+\frac{P_1r_3^2}{r_1}+\text{etc.}$$

$$=\frac{P_1}{r_1}(r_1^2+r_2^2+r_3^2, \text{ etc.})$$

$$=\frac{P_1}{r_1}\Sigma r^2$$

$$\text{or} \quad P_1=\frac{Mr_1}{\Sigma r^2}.$$

The direct shear will be divided equally between the bolts, i e.

$$S_1=S_2, \text{ etc.}=\frac{F}{\text{No. of bolts}},$$

the resultant shear, as before, being found by compounding the direct and torque shears.

The tabulation is as below:

Bolt.	r (in.).	r^2	Torque Shear $P=\frac{Mr}{\Sigma r^2}$ (lb.).	Direct Shear S (lb).	Resultant Shear R (graphically) (lb.).

Example 32.—Bolt Group.

Find in which bolt the maximum shear occurs in the bolt group shown in Fig. 90.

Bolts: all $\frac{1}{4}$-in diameter.

Applied Moment, $M = 2250$ lb in.
,, Shear, $F = 699$ lb.

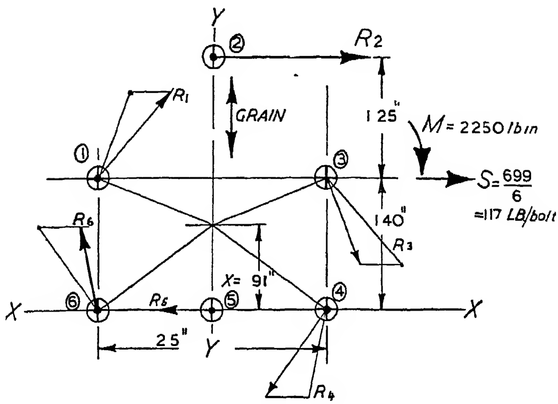

FIG 90

Position of C.G. ·

Since the bolts are symmetrically spaced about the vertical axis, the C.G. must lie on this line.

Taking moments about XX (A = area of bolt),

$$A \times 2\ 65 + 2A \times 1\ 4 = 6A\bar{x}$$

$$2{\cdot}65 + 2{\cdot}8 = 6\bar{x}$$

$$\bar{x} = \frac{5{\cdot}45}{6} = {\cdot}91 \text{ in.}$$

TABLE XXIV —SHEAR IN BOLTS (EXAMPLE 32).

Bolt	r (in.)	r^2.	$P = \frac{Mr}{\Sigma r^2}$ (lb)	S (lb)	Resultant R (lb.).	Angle θ to Grain (deg.)
1	1 35	1 82	247	117	310	42
2	1·74	3 02	318	,,	435	90
3	1·35	1 82	247	,,	310	42
4	1·55	2 40	283	,,	230	12
5	·91	83	170	,,	53	90
6	1 55	2 40	283	,,	230	12

$\Sigma r^2 = 12\ 29$

From Table XXIV it will be seen that the worst shear is in bolt (2) = 435 lb.

Bearing Strength of Bolts in Wooden Members.

The bearing strength of wood depends on the direction which the load makes to the grain, and curves giving the allowable bearing load perpendicular to and parallel to the grain for various thicknesses of wooden members are usually available.

If the direction of the load is at some intermediate angle to the grain (θ), then the allowable bearing load N at this angle to the grain

$$=\frac{PQ}{P\sin^2\theta+Q\cos^2\theta},$$

where P = allowable bearing load parallel to grain, and
Q = allowable bearing load perpendicular to grain.

If the bolt is loaded on one side only of the wooden member, as often happens, half the allowable values are used.

Example 33.—Bearing Strength of Bolts in Spruce.

In the previous example on an eccentrically loaded bolt group. the resultant load on each bolt and its direction to the grain has been given in Table XXIV. Assuming width of wooden spruce member = 2 0 in., to find the allowable bearing loads, assuming bolts are loaded on one side, given:

$$P=1700 \text{ lb.}$$
$$Q=\ 780 \text{ lb.}$$

The tabulation is as follows:—

TABLE XXV.—BEARING STRENGTH OF BOLTS IN SPRUCE (EXAMPLE 33).

Bolt.	Angle to Grain θ (deg.).	$\sin\theta$	$\sin^2\theta$.	$\cos\theta$.	$\cos^2\theta$.	$\frac{N}{2}=\frac{1}{2}\frac{PQ}{P\sin^2\theta+Q\cos^2\theta}$	Actual Load (lb.).	R.F.
1	42	·6691	·447	7431	552	$\frac{1326\times10^3}{1191}=556$	310	1 79
2	90	..	.	..	..	$\frac{780}{2}=390$	435	0 896
3	42	·6691	447	·7431	552	556	310	1 79
4	12	·2079	0432	9781	958	$\frac{1326\times10^3}{822}=808$	230	3 51
5	90	..	..	..	..	390	53	> 5
6	12	·2079	0432	9781	958	808	230	3 51

Bolt (2) is down in strength and would have to be increased to, say, $\frac{5}{16}$-in. dia. The bolt group would then have to be reworked, using the method for bolts of different diameter.

Example 34.—Combined Bending, Torsion, Shear and End Load on Engine-Bearer—Light Alloy Casting D.T.D. 289. (Fig. 91.)

$$\left.\begin{array}{lcl} X \text{ load} & = & 186 \text{ lb.} \\ Y \quad ,, & = & 520 \text{ lb.} \\ Z \quad ,, & = & 1050 \text{ lb} \end{array}\right\} \text{as shown.}$$

(*Note*—The minimum R F. for a casting must be 2 0)

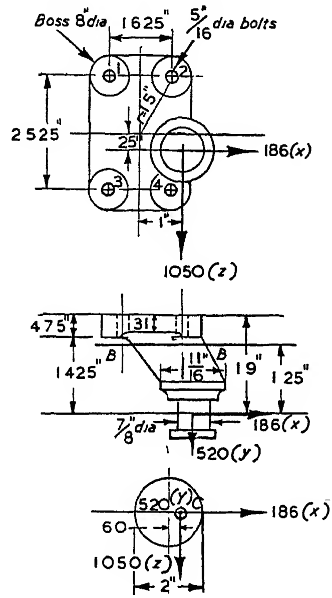

FIG 91.—Detail of engine-bearer

Loads in $\frac{5}{16}$-in. Bolts.

From "Z" load:

$$\text{Direct shear} = \frac{1050}{4} = 263 \text{ lb.}$$

Torque shear—

$M = 1050 \times 1 = 1050$ lb. in.;

$$r^2 = 1{\cdot}5^2; \quad P = \frac{Mr}{\Sigma r^2} = \frac{1050 \times 1{\cdot}5}{4 \times 1{\cdot}5^2} = 175 \text{ lb.}$$

This is a special case of an eccentric load on a bolt group.

Due to offset—

$$\text{Tension in (1)/(2)} = \tfrac{1}{2} \times \frac{1050 \times 1{\cdot}425}{2{\cdot}525} = 296 \text{ lb.}$$

Compression in (3)/(4) = 296 lb.

From "X" load: FACTOR

Direct shear $= \frac{186}{4} = 47$ lb.

Torque shear $M = 186 \times \cdot 25 = 47$ lb. in.

$$P = \frac{Mr}{\Sigma r^2} = \frac{47 \times 1\cdot 5}{4 \times 1\cdot 5^2} = \frac{47}{6} = \underline{8 \text{ lb.}}$$

Due to offset—

Tension in (1)/(3) $= \frac{1}{2} \times \frac{186 \times 1\cdot 425}{1\cdot 625} = 82$ lb.

Compression in (2)/(4) $= \underline{82 \text{ lb.}}$

From "Y" load:

Direct tension $= \frac{520}{4} = 130$ lb. each.

Due to offset—

Tension in (2)/(4) $= \frac{1}{2} \times \frac{520 \times 1}{1\cdot 625}$ 160 lb.

Compression in (1)/(3) = 160 lb.

Tension in (3)/(4) $= \frac{1}{2} \times \frac{520 \times \cdot 25}{2\cdot 525} = 26$ lb.

Compression (1)/(2) = 26 lb.

Maximum shear graphically = 400 lb. at (2). $\frac{5}{16}$-in. bolt at 3910 lb. gives > 5

Bearing strength in bearer
$= \cdot 475 \times \cdot 3125 \times 29{,}400 = \underline{2180 \text{ lb.}}$: at 400 lb.

Maximum Tension in Bolts—

(1)	(2)	(3)	(4)
296 (T.)	296 (T.)	296 (C.)	296 (C.)
82 (T.)	82 (C.)	82 (T.)	82 (C.)
130 (T.)	130 (T.)	130 (T.)	130 (T.)
160 (C.)	160 (T.)	160 (C.)	160 (T.)
26 (C.)	26 (C.)	26 (T.)	26 (T.)
508	636		
186	108		
322 (T.)	528 (T.)		

Allowable tension in $\frac{5}{16}$-in. bolt = 3960 lb. > 5

At section *BB*, we have combined bending, torque, shear and end load.

RESERVE FACTOR

Direct shear $=\sqrt{X^2+Z^2}$

$=\sqrt{186^2+1050^2}$

$=10^2\sqrt{3{\cdot}46+111}=1070$ lb.

Area $=\dfrac{\pi}{4}\times 2^2=\underline{\pi \text{ in.}^2}$

Maximum direct shear stress $=\dfrac{4}{3}$ Mean $=\dfrac{4}{3}\times\dfrac{1070}{\pi}=454$ lb./in.

Direct tensile stress $\dfrac{P}{A}=\dfrac{520}{\pi}=166$ lb./in.^{2}.

Bending stress—

B.M. due to X load $=186\times 1{\cdot}25=232$ lb. in.

B.M. due to Y load $=520\times{\cdot}60=312$ lb in.

Difference, M_{XY} $=80$ lb. in.

B M. due to Z load, $M_Z=1050\times 1{\cdot}25=1310$ lb. in.

Net $M=\ M_{XY}{}^2+M_Z{}^2$

$=\sqrt{80^2+1310^2}$

$=10^2\sqrt{8{\cdot}3+172}=\underline{1320 \text{ lb. in.}}$

$Z=\dfrac{\pi D^3}{32}\quad{\cdot}785$ in.^{3}.

Bending stress $=\dfrac{1320}{{\cdot}785}=\underline{1680 \text{ lb./in.}^2}$

Torque from "Z" load $=1050\times{\cdot}60=630$ lb. in.

Torque shear stress $=\dfrac{630}{2\times Z}=\dfrac{630}{1{\cdot}570}=402$ lb./in.^{2}

Net direct and bending stress—

$$p=\frac{P}{A}+\frac{M}{Z}=166+1680$$

$$=1846 \text{ lb./in.}^2$$

Principal direct stress $=\dfrac{1846}{2}+\sqrt{923^2+402^2}$

$=923+10^2\sqrt{85{\cdot}1+16{\cdot}1}$

$=923+1000$

$=\underline{1923 \text{ lb./in.}^2}$

D.T.D. 289 at 19,500 lb./in.^{2} $\quad > \underline{5}$

Maximum shear stress $=1000$ lb /in.^{2} at 13,500 lb./in.^{2} $\quad > 5$

(It will be noticed that the torque shear stress only has been used here in finding the principal direct stress. This is because the maximum direct shear stress and bending stress do not occur at the same point: the former is a maximum at the N.A. and the latter at the outermost fibre.)

Example 35.—Stability of a Lever.

It is often necessary to check the stability of a lever (or flat plate generally) and the following is an approximate method that can be adopted.

A load of 200 lb. acts on a lever (Specification 12G. L.3) as shown (Fig. 92), and it is required to check the stability along the compression side. In other words, it is desired to know whether this edge will buckle as a strut under load, even though the strength of the lever in other respects, such as bending, bursting, etc., is satisfactory.

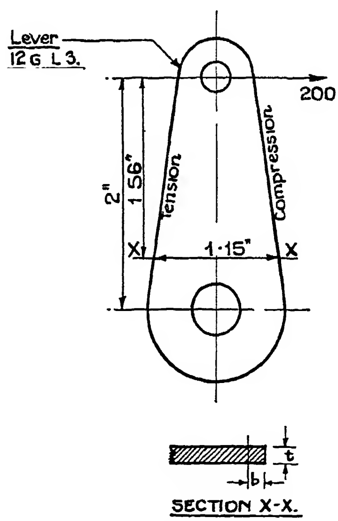

FIG. 92.—Detail of lever.

Consider a depth b on the compression side.

$$\text{Area } A = bt, \quad \text{where } t = \text{gauge of plate};$$

and

$$I = \frac{bt^3}{12}.$$

$$\therefore\ k^2 = \frac{I}{A} = \frac{bt^3}{12} \cdot \frac{1}{bt} = \frac{t^2}{12}$$

$$k = \frac{t}{\sqrt{12}} = \frac{\cdot 104}{3{\cdot}47} = \cdot 03 \text{ in this case.}$$

$$\frac{l}{k} = \frac{2}{\cdot 03} = 67.$$

RESERVE FACTOR

Allowable stress for T.4 (from strut curve in the chapter on "Struts")

$$=17{,}300 \text{ lb./in.}^2$$

This figure must not be exceeded by the actual bending stress at any section such as XX. At XX,

$$Z = \cdot 104 \times \frac{1\cdot 15^2}{6} = \cdot 023 \text{ in.}^3$$

$$\frac{M}{Z} = \frac{200 \times 1\cdot 56}{\cdot 023} = 13{,}600 \text{ lb./in.}^2 : \text{ as a strut} \qquad 1\cdot 27$$

Allowable bending stress at 39,000 lb./in.² 2·87

Example 36.—Bending of Bolt at Fork End.

Load: 760 lb. applied to a $\frac{3}{16}$-in. S.1 bolt as shown (Fig. 93).

As regards the bolt itself, it would be over-pessimistic to consider that we have a concentrated load of 760 lb. applied at the centre.

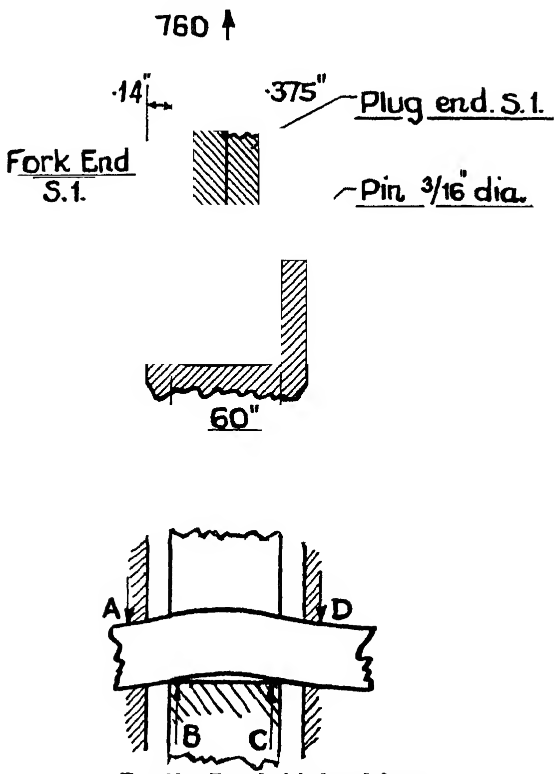

FIG. 93.—Detail of fork end fitting

RESERVE FACTOR

Shown exaggerated, the bolt will bend as in the lower part of Fig. 93, and we can assume, therefore, that we have an applied load of 380 lb. at B and C, and reactions of 380 lb. at A and D.

The point of application of the load at B and C is estimated thus Assume that the load at B, say, acts over a bearing surface equal to $\frac{1}{8}$ the width of the plug-end, i.e. $\frac{\cdot 375}{8} = \cdot 046$ in.

Then bearing stress at B

$$\frac{380}{\cdot 1875 \times \cdot 046} = 4310 \text{ lb./in.}^2 \qquad \text{S.1 at } 117{,}000 \text{ lb./in.}^2 \qquad > 5$$

The assumption as to bearing surface is thus satisfactory.

As regards the reactions at A and D, the same assumption can be made.

The loads on the pin are then located as in Fig. 94.

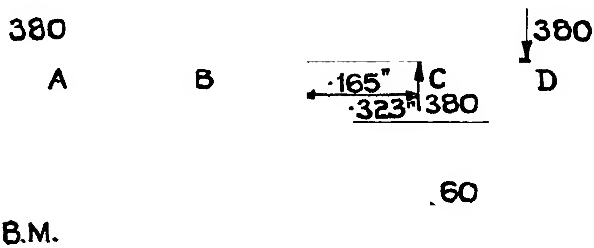

FIG. 94.—Location of loads on pin

$$\text{Maximum B.M.} = 380 \times \cdot 323 - 380 \times \cdot 165 = 60 \text{ lb. in.}$$

$$Z = \frac{\pi}{32} \cdot D^3 = \frac{\pi}{32}(\cdot 1875)^3 = \cdot 000653 \text{ in.}^3$$

$$\frac{M}{Z} = \frac{60}{\cdot 000653} = 91{,}800 \text{ lb./in.}^2 \qquad \begin{array}{ll} \text{S.1 at } 69{,}000 \text{ lb./in.}^2 & 0\cdot 75 \\ \text{S.11 at } 110{,}000 \text{ lb./in.}^2 & 1\cdot 20 \end{array}$$

It is therefore necessary to use an S.11 bolt.

If we had assumed a concentrated load of 760 lb. at the centre of the bolt, the maximum B.M. would have been

$$Wl \quad 760 \times \cdot 60 \quad 101 \text{ lb. in.,}$$

that is, almost double the value we have used.

Example 37.—Distribution of Bending Moment at Joint.

The type of problem illustrated (Fig. 95) occurs very often in engine mountings and undercarriages.

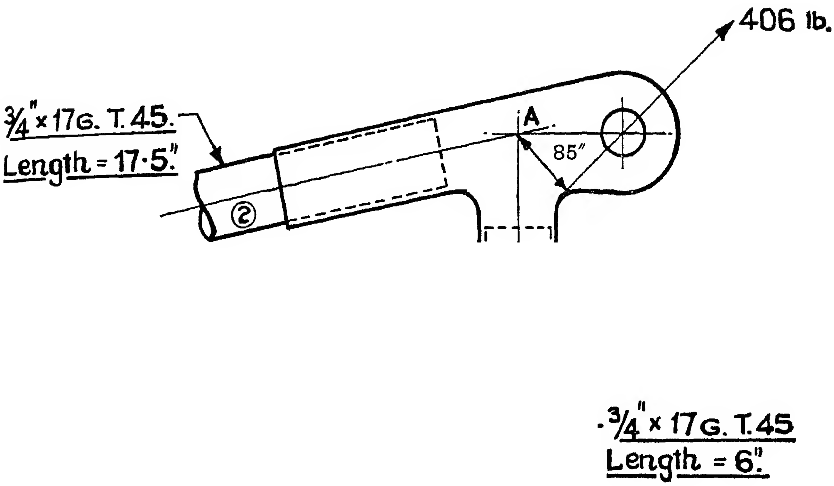

FIG. 95.—Detail of engine-mounting joint.

The Bending Moment applied to the joint at A

$$=406 \times \cdot 85 = \underline{345 \text{ lb. in.}}$$

This will be shared between tubes (2) and (3), and as an approximation we can consider that this will be in proportion to their $\frac{EI}{L}$ ratios, since this ratio is a measure of the stiffness of the tubes.

For $\frac{3}{4}$-in. 17G. tube, $I = \cdot 0074$ in.^{4}.

$$\left.\begin{aligned} &\text{For tube (2), } \frac{I_2}{L_2} = \frac{\cdot 0074}{17 \cdot 5} = \cdot 00042 \\ &\;\;,, \quad ,, \quad (3), \frac{I_3}{L_3} = \frac{\cdot 0074}{6} = \cdot 00124 \end{aligned}\right\} \frac{I_2}{L_2} + \frac{I_3}{L} = 00166$$

$$\therefore \text{ Proportion of B.M. in (2)} = \frac{\cdot 00042}{\cdot 00166} \times 345 = \underline{88 \text{ lb. in.}}$$

(since E is the same for both tubes)

$$\text{and in (3)} = 345 - 88 = \underline{257 \text{ lb. in.}}$$

Knowing the end load in (2) and (3), these members could now be stressed.

Example 38.—Plate Fitting.

RESERVE FACTOR

Fitting acted on by load of 10,000 lb. as shown (Fig. 96).

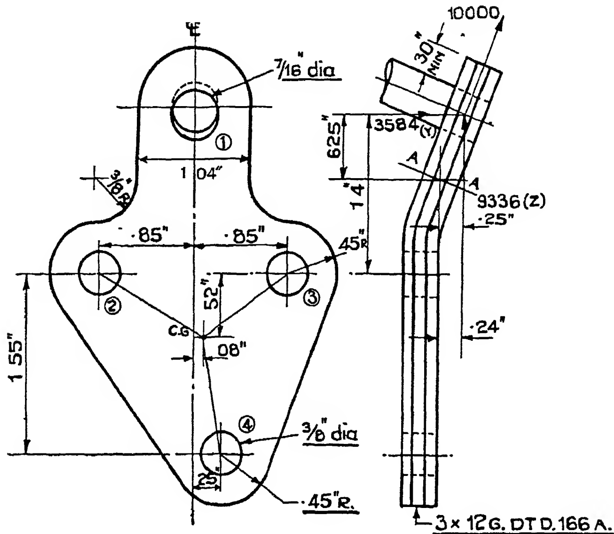

FIG. 96 —Detail of fitting.

Bolt (1).—Shear load = 10,000 lb.

Allowable in $\frac{7}{16}$-in. S.1 = 7650 lb. — 0·76

" " $\frac{7}{16}$-in. S.2 = 11,850 lb. — 1·18

A $\frac{7}{16}$-in. S.2 bolt must therefore be used.

Bearing strength of 3 × 12G. D.T.D. 166A—

$\cdot 4375 \times 3 \times \cdot 104 \times 174{,}000 = 23{,}800$ lb. at 10,000 lb. — 2·38

Bursting strength—

$1\cdot 75 \times \cdot 312 \times \cdot 30 \times 72{,}000 = 11{,}800$ lb. at 10,000 lb — 1·18

Components of the 10,000-lb. load are: 9336 lb. upward (Z).

3584 lb. outward (Y).

Shear in Bolts (2), (3), (4).—The position of C.G. is shown in Fig. 96, and the resultant shear in each bolt worked out in Table XXVI.

$$M = 9336 \times \cdot 08 = 746 \text{ lb. in. and}$$
$$M/\Sigma r^2 = 232.$$

TABLE XXVI.—SHEAR IN BOLTS (EXAMPLE 38)

Bolt.	r.	r^2	$P=232r$	Angle with Centre Line	Components Z	Components Y.	Direct Z	Net Z	Net Y.	Resultant (lb.)
2	1·10	1·21	255	30°	−221	−128	+3112	+2890	−128	2900
3	0·95	0·90	221	36°	+184	−130	+3112	+3296	−130	3300
4	1·05	1·10	244	81°	+ 40	+241	+3112	+3152	+241	3160

$\Sigma r^2 = 3·21$

Tension in bolts (2), (3), (4) RESERVE FACTOR

From 9336 lb. (Z)—

$$\text{Tension in (4)} = \frac{9336 \times 24}{1·55} = 1450 \text{ lb}$$

$$\text{Compression in (2) or (3)} = \tfrac{1}{2} \,.\, 1450 = 725 \text{ lb.}$$

From 3584 lb. (Y)—

$$\text{Tension in (2) or (3)} = \tfrac{1}{2} \,.\, \frac{3584 \times 2·95}{1·55} = 3400 \text{ lb.}$$

$$\text{Net tension in (2) or (3)} = 2675 \text{ lb}$$

$$\text{Tensile stress } p \text{ in } \tfrac{3}{8}\text{-in. bolt } \frac{2675}{·1105} = 24{,}200 \text{ lb./in.}^2$$

$$\text{Shear stress in bolt (3)} = \frac{3300}{·1105} = 29{,}900 \text{ lb./in.}^2$$

Principal direct stress

$$= 12{,}100 + 10^3\sqrt{12·1^2 + 29·9^2}$$

$= 12{,}100 + 32{,}300 = 44{,}400$ lb./in.^{2}. S 1 at 78,000 lb./in.^{2}. <u>1·76</u>

Maximum shear stress

$= 32{,}300$ lb./in.^{2}: S.1 at 51,000 lb./in.^{2}. <u>1·58</u>

Bearing in 3×12G D.T.D. 166A at (3)—

Strength $= ·375 \times ·312 \times 174{,}000 = 20{,}400$ lb.: at 3300 lb. > 5

Bending at Section AA— RESERVE FACTOR

$$M = 3584 \times \cdot 625 - 9336 \times \cdot 25$$
$$= 2240 - 2340 = 100 \text{ lb. in.}$$

$$Z = 1\cdot 04 \times \frac{\cdot 312^2}{} = \cdot 0169 \text{ in.}^3$$

$$\frac{M}{Z} = \frac{100}{\cdot 0169} = 5920 \text{ lb./in.}^2$$

$$\frac{P}{A} = \frac{10{,}000}{\cdot 312 \times 1\cdot 04} = 30{,}800 \text{ lb./in.}^2$$

$$\frac{P}{A} + \frac{M}{Z} = 36{,}720 \text{ lb./in.}^2\text{: D.T.D. 166A at } 116{,}000 \text{ lb./in.}^2$$ 3·16

Example 39.—Hinge Bracket: D.T.D. 300, Casting.

Loads applied as shown (Fig. 97).

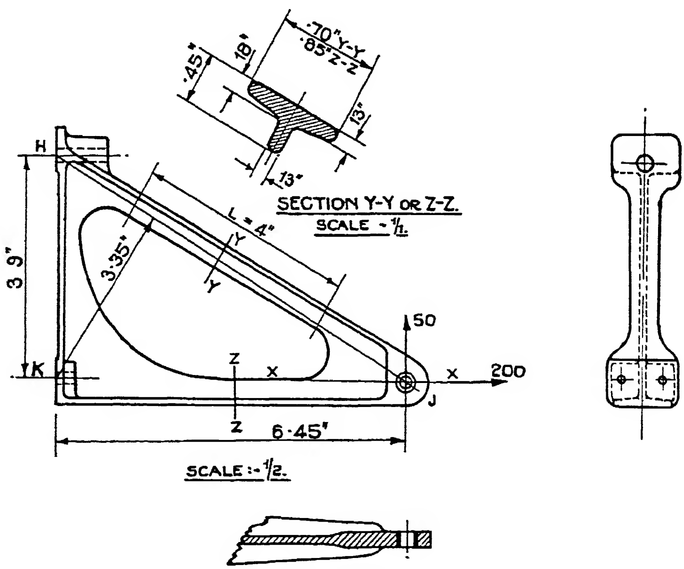

FIG. 97.—Detail of casting.

Member JK:

By moments about H,

$$P_{JK} \times 3\cdot 9 = 200 \times 3\cdot 9 + 50 \times 6\cdot 45$$
$$P_{JK} = 282 \text{ lb. (tension).}$$

Mean area at section ZZ (see Fig 98). RESERVE FACTOR

$$= \cdot 13 \times \cdot 45 + \cdot 72 \times \cdot 15 = \cdot 0586 + \cdot 108 = \cdot 167 \text{ in}^2$$

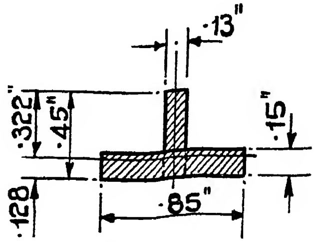

FIG. 98 —Approximate mean section at ZZ.

Position of N.A.—

$$\cdot 0586 \times \cdot 225 + \cdot 108 \times \cdot 075 = \cdot 167 \bar{x}$$

$$\cdot 0132 + \cdot 0081 = \cdot 167 \bar{x}$$

$$\bar{x} = \frac{\cdot 0213}{\cdot 167} = \cdot 128 \text{ in.}$$

I_{NA}—

$$\cdot 0586 \times \frac{\cdot 45^2}{12} = \cdot 00099$$

$$\cdot 0586 \times \cdot 097^2 = \cdot 00055$$

$$\cdot 108 \times \frac{\cdot 15^2}{12} = \cdot 00020$$

$$\cdot 108 \times \cdot 053^2 = \cdot 00030$$

$$\overline{\cdot 00204 \text{ in.}^4}$$

$M = 282 \times \cdot 322 = 90 \cdot 6$ lb. in. (since distance from line of application of load to NA $= \cdot 322$ in.).

$$Z = \frac{\cdot 00204}{\cdot 322} = \cdot 00633 \text{ in.}^3$$

$$\frac{M}{Z} = \frac{90 \cdot 6}{\cdot 00633} = 14{,}300 \text{ lb./in.}^2$$

$$\frac{P}{A} = \frac{282}{\cdot 167} = 1{,}690 \text{ lb./in.}^2$$

$\dfrac{P}{A} + \dfrac{M}{Z} = 15{,}990$ lb./in.^{2}: D.T.D. 300 at 17,900 lb./in.^{2} 1·12

This result is in any case pessimistic, since it neglects the fixing at the ends of JK due to the web.

Member JH. RESERVE FACTOR

$$P_{JH} \times 3{\cdot}35 = 50 \times 6{\cdot}45$$

$$_{JH} = 96{\cdot}2 \text{ lb. (compression).}$$

Rough check as a strut:

$$k^2 = \frac{I}{A} = \frac{\cdot00204}{\cdot167} = \cdot0122 \text{ (assuming same section at } YY \text{ as at } ZZ).$$

$$k = \cdot1105.$$

Taking $l = 4$,

$$\frac{l}{k} = \frac{4}{\cdot1105} = 36,$$

and the corresponding allowable stress for D.T.D. 300 = 13,000 lb./in.^{2}.

Allowable load in $JH = 13{,}000 \times \cdot167 = \underline{2170 \text{ lb.}}$: at 96·2 lb. 2·26

Example 40.—Strength of Weld.

Loads act as shown (Fig. 99).

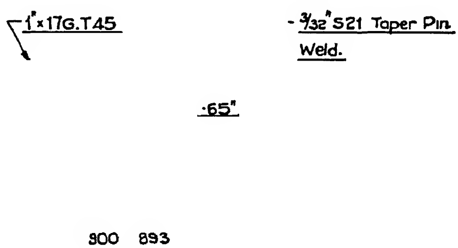

FIG. 99.—Detail of socket.

Section YY:

$$M = 900 \times \cdot77 + 893 \times \cdot37$$
$$= 693 + 331 = 1024 \text{ lb. in.}$$
$$Z \text{ for 1 in.} \times 17G. = \cdot0371 \text{ in.}^3$$
$$\frac{M}{Z} = \frac{1024}{\cdot0371} = 27{,}600 \text{ lb./in.}^2. \quad \text{T.45 at } 100{,}000 \text{ lb./in.}^2.$$ 3·63

Shear load on Taper Pin (neglect weld for the time being)

$$\frac{1024}{1{\cdot}0} = 1024 \text{ lb.}$$

$\frac{3}{32}$-in. S.21 Taper Pin at 250 lb. is therefore clearly down in strength if the weld is neglected.

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Weld at Z.

Bending stress at $Z = 27{,}600$ lb./in.^{2} (say). Consider 1 in. around the circumference at Z.

$$\text{Area} = \cdot 056 \times 1 = \cdot 056 \text{ in}^2.$$

$$\text{Load per inch} = 27{,}600 \times \cdot 056 = \underline{1545 \text{ lb.}}$$

Allowable shear load per inch for 17G. T.45 = 1680 lb. 1·09

Joint is therefore considered satisfactory

Weld at X:

There will be very little load on the weld at X. Load will be transferred from the inner tube to the socket as a bearing load.

Example 41.—Welded Joint.

Loads: 330 lb (Z) and 690 lb. (X) applied as shown (Fig. 100).

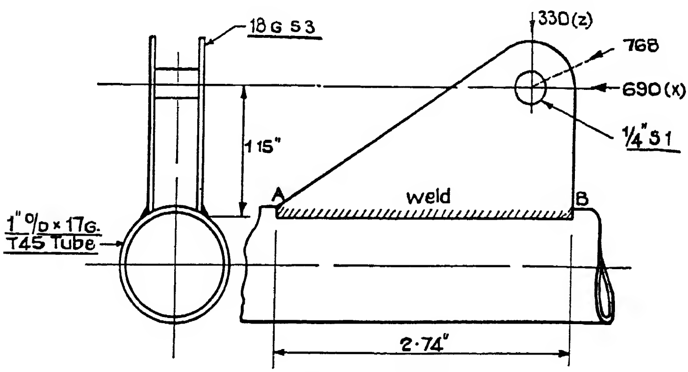

FIG. 100.—Detail of welded joint

Shear load on weld per side = 690

Load per inch = $\dfrac{690}{2 \times 2{\cdot}74} = 126$ lb. 18G S.3 at 962 lb. > 5

Bending stress at B per side:

$$M = \frac{690}{2} \times 1{\cdot}15 = 397 \text{ lb. in.}$$

$$Z = \cdot 048 \times \frac{2{\cdot}74^2}{6} = \cdot 06 \text{ in.}^3$$

$$\frac{M}{Z} = \frac{397}{\cdot 06} = \underline{6620 \text{ lb./in.}^2}$$

Considering 1 in. along weld at B, RESERVE FACTOR

$$\text{Area} = \cdot 048 \times 1 = \cdot 048 \text{ in.}^2$$

$$\text{Load per inch} = \cdot 048 \times 6620 = \underline{317 \text{ lb.}} \text{ (tension).}$$

This will be relieved by direct compression.

$$\text{Direct compression per inch} = \frac{330}{2 \times 2 \cdot 74} = 60 \text{ lb.}$$

$$\therefore \text{ Net tension} = 317 - 60 = \underline{257 \text{ lb./in.}},$$

which is considered satisfactory.

Example 42.—Bearing Strength of Bracket on Spar Face.

Due to the offset load of 300 lb., the bracket will bear against the spar face from A to B.

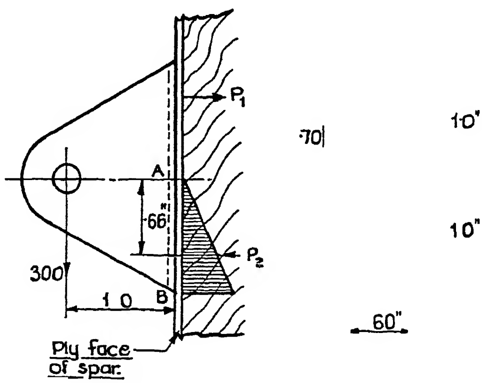

FIG. 101 —Detail of bracket

If we assume that there is a triangular distribution of bearing stress (as shown shaded), we can assume as a fair approximation that the moment is resisted by a force P_2 acting at 2/3 AB, together with an equal tension P_1 in the upper bolt.

$$\text{Then } P_2 = \frac{300 \times 1 \cdot 0}{1 \cdot 36} = 221 \text{ lb.}$$

$$\text{Bearing area of spar face} = 1 \cdot 0 \times \cdot 60 = \cdot 60 \text{ in.}^2$$

$$\text{Mean bearing stress} = \frac{221}{\cdot 60} = 368 \text{ lb./in.}^2$$

Maximum bearing stress, which is $2 \times$ mean for triangular distribution,

$$= 2 \times 368 = \underline{736 \text{ lb./in.}^2}.$$

Taking the allowable bearing stress of ply as 1000 lb./in.² 1·36

ERVE FACTOR

Example 43.—Edge Stress at a Bend.

Lever as shown (Fig. 102) B.M. at section $XX = \underline{700 \text{ lb in.}}$

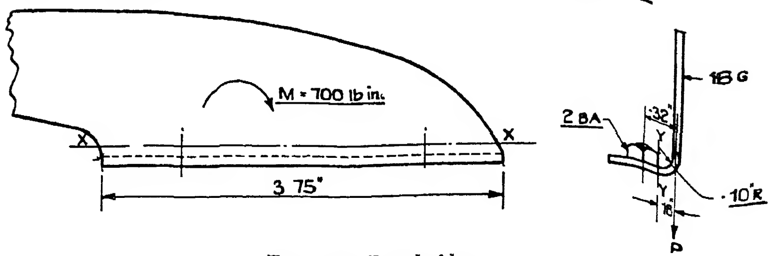

Fig. 102.—Detail of lever

At section XX ·

$$Z = \cdot064 \times \frac{3\cdot75^2}{6} = \cdot151 \text{ in.}^3$$

$$\text{Bending stress} = \frac{700}{\cdot151} = \underline{4640 \text{ lb /in.}^2}$$

Consider a length of 1 in. along XX.

$$\text{Area} = \cdot064 \times 1 = \cdot064 \text{ in}^2$$

$$\text{Load per inch } P = 4640 \times \cdot064 = 297 \text{ lb.}$$

At section YY

$$\text{Moment of } P \text{ about } YY = 297 \times \cdot16 = 47\ 5 \text{ lb. in.}$$

$$Z \text{ per inch along } YY \quad = 1\cdot0 \times \frac{\cdot064^2}{6} = \cdot0007 \text{ in.}^3$$

$$\text{Bending stress} \quad = \frac{47\cdot5}{0007} = \underline{67{,}800 \text{ lb./in.}^2}$$

S.3 at 45,500 lb./in.² $\underline{0\cdot67}$

D.T.D. 166A at 103,000 lb./in.² $\underline{1\cdot52}$

If S.3 is used, it will be necessary to use a profile washer as shown (Fig. 103).

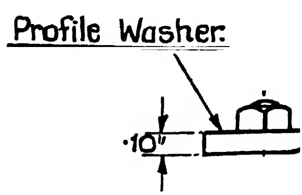

Fig 103.—Method of using a profile washer

Try washer ·10 in. thick S.1. RESERVE FACTOR

$$Z \text{ per inch along } YY \text{ of washer plus plate} = 1{\cdot}0 \times \frac{{\cdot}164^2}{6} = {\cdot}00447 \text{ in.}^3$$

$$\text{Bending stress } \frac{47{\cdot}5}{{\cdot}00447} = 10{,}600 \text{ lb./in.}^2. \qquad \text{In S.3.}: \qquad 4{\cdot}3$$

Example 44.—Beam with Offset Load.

Load of 768 lb acting as shown (Fig 104) on beam simply supported at A and B

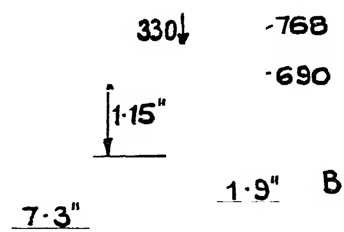

FIG. 104 —Offset load on beam

The inclined load of 768 lb. can be resolved into 330 lb. normal and 690 lb. parallel to AB.

This load of 690 lb. is then equivalent to an end load in AB of 690 lb., probably all reacted at A or B, depending on the end fixing, plus a moment equal to $690 \times 1{\cdot}15 = 793$ lb. in.

Reactions at A and B:

(a) Due to lateral load of 330 lb.—

$$R_B = 330 \times \frac{5{\cdot}4}{7{\cdot}3} = 244 \text{ lb. (upward).}$$

$$R_A = 330 - 244 = 86 \text{ lb. (upward).}$$

(b) Due to moment of 793 lb. in.—

This will be resisted by the couple formed by an upward reaction at A (and a downward reaction at B) multiplied by the arm AB, i e.

$$R_B = \frac{793}{7{\cdot}3} = 108{\cdot}5 \text{ lb. (downward).}$$

$$R_A = 108{\cdot}5 \text{ lb. (upward).}$$

Net reactions:

$$R_A = 194{\cdot}5 \text{ lb. (upward).}$$

$$R_B = 135{\cdot}5 \text{ lb. (downward).}$$

Bending Moment Diagram.

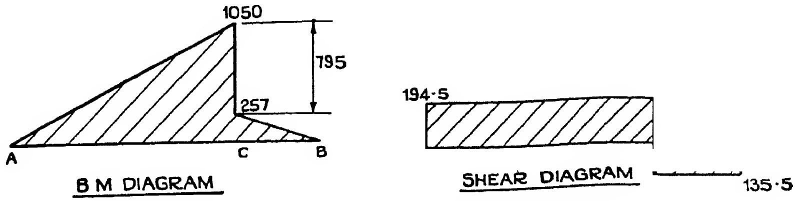

FIG 105.—Bending moment and shear diagrams

B.M. at C:

Moment of forces to left of $C = 194{\cdot}5 \times 5{\cdot}4 = \underline{1050 \text{ lb. in.}}$

„ „ „ right „ $= 135{\cdot}5 \times 1{\cdot}9 = \underline{257 \text{ lb. in.}}$

As a check, the difference between ordinates at C on B.M. diagram $= 1050 - 257 = 793$ lb. in. must equal the applied offset moment at C.

Example 45.—Differential Bending of Lever.

Fig. 106 shows a special type of lever with a load P acting as indicated.

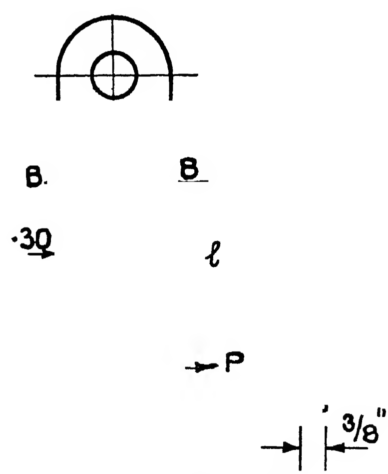

FIG. 106 —Detail of lever.

If the arms AB were not interconnected at AA, each would bend as shown exaggerated at (*a*) (Fig. 107), but, owing to their interconnection, there will be a fixing moment applied at AA, and the actual bending can be considered as that at (*b*). There is thus a

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point of contraflexure, and therefore zero bending moment, at C, the mid-point of AB, the net B.M. being as shown shaded at (*b*) (Fig. 107).

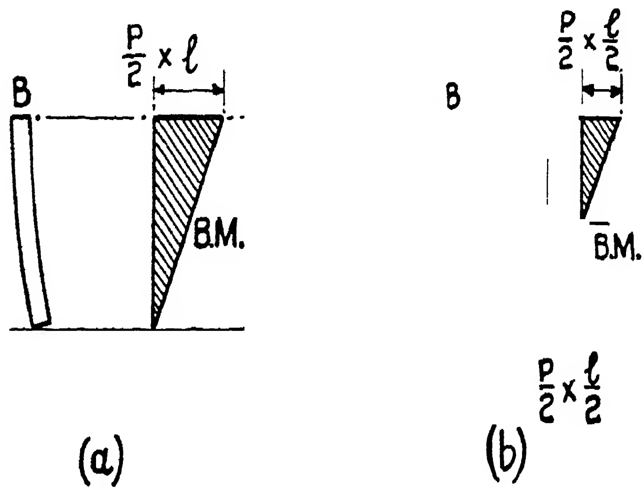

FIG. 107.

Thus, at B we have halved the B.M. from $\frac{P}{2}\times l$ to $\frac{P}{2}\times\frac{l}{2}$, but at A have a fixing moment $=\frac{P}{2}\times\frac{l}{2}$.

Taking $P=400$ lb. and $l=2$ in.,

$$M=\frac{P}{2}\times\frac{l}{2}=200 \text{ lb. in.}$$

$$\cdot375\times\cdot30^2 \qquad \cdot00562 \text{ in.}^3$$

$$f=\frac{200}{\cdot00562}=\underline{35{,}600 \text{ lb./in.}^2} \qquad \text{L.3 at } 39{,}000 \text{ lb./in.}^2 \qquad 1\cdot1$$

PART III.

STRAIN ENERGY.

WHEN a member is subjected to, say, an end load, work is done in extending or compressing it and, provided the stress developed does not exceed the elastic limit of the material, practically the whole of the strain will disappear when the load is removed. Whilst under load, therefore, the work done in straining the material is stored as potential or strain energy, which is denoted by $\bar{u}$.

Strain Energy due to an End Load (Gradually Applied).

Consider a tensile load P applied to a member of length L such that the extension is x.

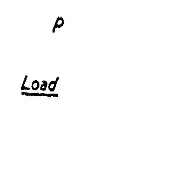

FIG 108 — Load/extension diagram

From the load/extension diagram (Fig. 108), the work done is seen to be the area under the curve, i.e. $\frac{1}{2}Px$, but since $E=\frac{\text{Stress}}{\text{Strain}}=\frac{PL}{Ax}$, where E = Young's Modulus and A = the cross-sectional area,

$$x=\frac{PL}{AE}.$$

That is, the energy stored (strain energy)—

$$=\frac{P^2L}{2AE}.$$

In a similar way it can be proved that the strain energy due to bending (gradually applied) is $\int_0^L \frac{M^2}{2EI}\,dx$, where M is the Bending Moment, and that due to Torque (gradually applied) is $\frac{T^2L}{2GI_P}$, where T = the Torque, G = Torsional Modulus of Rigidity and I_P = the Polar Moment of Inertia.

Theorem.

The Partial Differential Coefficient of the total strain energy, expressed in terms of the external load system with respect to one of the external loads, is the movement of that load *in its line of action.*

That is, if

$\bar{u}$ = the total strain energy of a body acted on by forces P_1, P_2, P_3, . . . P_N, and

y_N = the deflection under the load P_N *in its line of action,*

$$y_N = \frac{\partial \bar{u}}{\partial P_N}$$

The application of this theorem will be shown by means of worked examples.

Example 46.—Simply Supported Beam with Concentrated Load.

Beam of span L, carrying a concentrated load W as shown (Fig. 109).

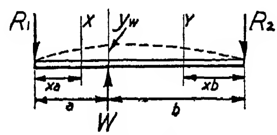

FIG. 109.

Reactions:

$$R_1 = W \cdot \frac{b}{a+b}, \qquad R_2 = W \cdot \frac{a}{a+b}.$$

At any section X between R_1 and W,

$$M_X = R_1 x_a = W \cdot \frac{b}{a+b} \cdot x_a.$$

At any section Y between R_2 and W,

$$M_Y = R_2 x_b = W \cdot \frac{}{a+b} \cdot x_b.$$

Total strain energy—

$$\bar{u} = \int_0^a \frac{{M_X}^2}{2EI}\,dx + \int_0^b \frac{{M_Y}^2}{2EI}\,dx$$

$$= \frac{1}{2EI}\left[\int_0^a \left(W \cdot \frac{b}{a+b} \cdot x_a\right)^2 dx + \int_0^b \left(W \cdot \frac{a}{a+b} \cdot x_b\right)^2 dx\right]$$

$$= \frac{W^2}{2EI(a+b)^2}\left[b^2 \int_0^a {x_a}^2\,dx + a^2 \int_0^b {x_b}^2\,dx\right]$$

$$= \frac{W^2}{2EI(a+b)^2}\left[b^2 \frac{a^3}{3} + a^2 \cdot \frac{b^3}{3}\right]$$

$$= \frac{W^2}{6EI} \cdot \frac{a^2 b^2}{a+b}$$

Deflection under load $W = y_W = \dfrac{\partial \bar{u}}{\partial W} = \dfrac{W}{3EI} \cdot \dfrac{a^2 b^2}{a+b}$.

Check.—For a load W at the centre, $a = b = L/2$ in the expression just given, and

$$y_W = \frac{W}{3EI} \cdot \frac{a^4}{2a} - \frac{Wa^3}{6EI} - \frac{WL^3}{48EI},$$

a standard form.

Above it has been stated that $\bar{u} = \int_0^L \dfrac{M^2}{2EI} dx$ and that $y_W = \dfrac{\partial \bar{u}}{\partial W}$.

It can be shown that these two expressions can be combined to give

$$y_W = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial W} dx,$$

which will be used in the worked example that follows.

Example 47.—Simply Supported Beam carrying Uniformly Distributed Load and Concentrated Load.

Beam of span $2L$ simply supported at A and B and carrying a uniformly distributed load of w per unit run and a concentrated load P at the centre.

Reactions:

$$R_A = R_B = wL + \frac{P}{2}.$$

At any section X, distant x from A,

$$\text{Bending moment } M_X = R_A \cdot x - \frac{wx^2}{2}$$

$$= \left(wL + \frac{P}{2}\right) x - \frac{wx^2}{2},$$

and

$$\frac{\partial M_X}{\partial P} = \frac{x}{2}.$$

Deflection at the centre—

$$y_P = \frac{\partial \bar{u}}{\partial P} = 2\int_0^L \frac{M_X}{EI} \cdot \frac{\partial M_X}{\partial P} dx.$$

[*Note.*—Since the beam is symmetrical about the centre line, the total strain energy will be twice that for each half.]

$$y_P = \frac{2}{EI} \int_0^L \left[\left(wL + \frac{P}{2}\right)x - \frac{wx^2}{2}\right] \frac{x}{2} dx$$

$$= \frac{1}{EI} \left[\left(wL + \frac{P}{2}\right)\frac{L^3}{3} - \frac{wL^4}{8}\right].$$

When $P=0$, that is, when there is a distributed load only,

$$y_P=\frac{1}{EI}\left[\frac{wL^4}{3}-\frac{wL^4}{8}\right]=\frac{5wL^4}{24EI}.$$

Check.—If we call the span L_1 instead of $2L$,

$$=\frac{5wL_1{}^4}{384EI},$$

a standard form.

Example 48.—Wheel Fork.

The loads acting are shown in Figs. 110 and 111.

$$\frac{4080}{2}\text{ lb.}=2040\text{ lb. per side,}$$

which resolves into 1880 lb. and 795 lb. per side as shown.

$$\begin{aligned}\text{From Fig. 110, } ED&=4-4\cos\theta-BE\\&=4(1-\cos\theta)-4{\cdot}1\sin 2\\&=4(1-\cos\theta)-{\cdot}14\\&=(3{\cdot}86-4\cos\theta)\end{aligned}$$

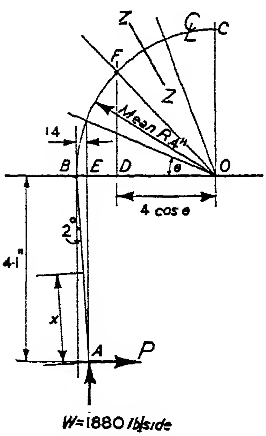

Fig. 110.—True view of wheel fork.

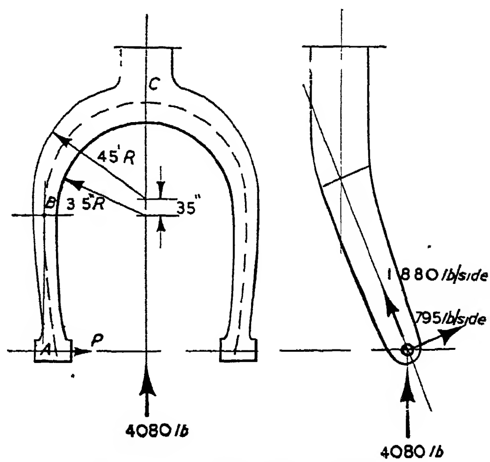

Fig. 111.—Detail of wheel fork

Expressions for bending moment about axis XX. (See Fig. 112.)

From A to B:

$$\begin{aligned}M&=Wx\sin 2+Px\cos 2\\&=W\times{\cdot}035x+P\times{\cdot}99x\\&=Wf_1(x)+Pf_2(x).\end{aligned}$$

From B to C:

$$M = -W \times ED + P(AE + DF)$$
$$= W(4 \cos \theta - 3{\cdot}86) + P(4{\cdot}1 + 4 \sin \theta)$$
$$= Wf_1(x) + Pf_2(x).$$
$$y_P = \frac{\partial \bar{u}}{\partial P} = \frac{1}{E}\int \frac{M}{I} \cdot \frac{\partial M}{\partial P} dx$$

(the expression is put in this form because I varies from section to section)

and $$M = Wf_1(x) + Pf_2(x),$$

so that $$\frac{\partial M}{\partial P} = f_2(x).$$

Substituting,

$$E\frac{\partial \bar{u}}{\partial P} = \int\left\{\left[\frac{W}{I}f_1(x) + \frac{P}{I}f_2(x)\right]f_2(x)\right\}dx$$
$$= W\int\frac{f_1(x)\,f_2(x)}{I} + P\int\frac{(f_2(x))^2}{I}.$$

The integration of this expression is carried out graphically. Since I is varying, find the moment of inertia at different stations (as in Table XXVII), the value of I for an ellipse being $\frac{\pi}{64}bd^3$ (see Fig. 112). Then find values of $f_1(x)$, $f_2(x)$, etc., as in Table XXVIII, plot $\frac{f_1(x)\,f_2(x)}{I}$ and $\frac{(f_2(x))^2}{I}$ against x (Fig. 113), and so obtain the area under these curves.

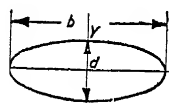

FIG. 112 —Elliptical section.

TABLE XXVII.—MOMENT OF INERTIA AT VARIOUS STATIONS.

Station.	$\theta°$.	b (in.).	d (in).	d^3.	$I = \frac{\pi}{64} bd^3$ (in.^{4}) for Ellipse
A	..	1 50	8	512	·0377
$\frac{1}{2}$ way A–B	..	1 80	85	612	054
B	0	2 125	1 025	1 07	112
$\frac{1}{2}$ way B–C	45	2 55	1·20	1 73	2075
C	90	2·55	1 25	1 95	244
	$22\frac{1}{2}$	2 35	1·125	1 38	·159
	$67\frac{1}{2}$	2 55	1 25	1 95	·244

TABLE XXVIII.—EVALUATION OF STRAIN ENERGY FUNCTIONS.

Station.	$\theta°$.	x (in.)	$f_1(x)$	$f_2(x)$.	I (in.^{4}).	$\frac{f_1(x)f_2(x)}{I}$.	$\frac{(f_2(x))^2}{I}$.
A	.	0	0	0	0377	0	0
$\frac{1}{2}$ way A–B	..	2 05	0 72	2·05	·054	2 74	77 5
B	0	4·10	·144	4·10	·112	5 27	150
..	$22\frac{1}{2}$	5 68	– ·17	5·63	·159	–6 02	199
..	45	7·08	–1·04	6·93	·2075	–34 7	231
..	$67\frac{1}{2}$	7·96	–2·34	7·80	244	–74 8	249
C	90	8 40	–3 87	8·10	244	–128 5	268

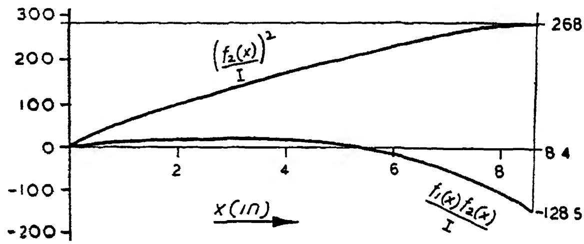

FIG. 113.—Graph of strain energy functions

$$\text{Area under } \frac{f_1(x)f_2(x)}{I} \text{ curve} = -100.$$

$$\text{Area under } \frac{(f_2(x))^2}{I} \text{ curve} \quad =1220.$$

$$\text{Therefore } E\frac{\partial \bar{u}}{\partial P} = W\int \frac{f_1(x)f_2(x)}{I} + P\int \frac{(f_2(x))^2}{I}$$

$$= -100W + 1220P.$$

But $E\frac{\partial \bar{u}}{\partial P}=0$, since there is no deflection in the direction of P.

$$\therefore\ P = \cdot 082W$$
$$= \cdot 082 \times 1880$$
$$= \underline{154 \text{ lb.}}$$

The bending moment about the X axis at any section ZZ is given by the expression:

$$M_{XX} = 1880 f_1(x) + 154 f_2(x),$$

and values of M_{XX} are worked out in Table XXIX.

TABLE XXIX.—BENDING MOMENT AT VARIOUS STATIONS.

(Bending moment at any section ZZ at right angles to axis of fork.)

$$M_{XX} = 1880 f_1(x) + 154 f_2(x).$$

Station.	$\theta°$	x (in.)	$1880 f_1(x)$	$154 f_2(x)$	M_{XX} (lb in.)
B	0	4 10	270	631	901
„	$22\frac{1}{2}$	5 68	− 319	866	547
„	45	7 08	−1950	1070	− 880
„	$67\frac{1}{2}$	7·96	−4400	1200	−3200
C	90	8·40	−7270	1250	−6020

Strength at Section B:

$$\text{Area of ellipse} = \frac{\pi bd}{4}$$

$$= \frac{\pi}{4} \times 2{\cdot}125 \times 1{\cdot}025$$

$$= 1{\cdot}71 \text{ in.}^2$$

$M_{XX} = 901$ lb. in. from Table XXIX.

$$M_{YY} = 795 \times 4{\cdot}1$$

$$= 3260 \text{ lb. in.}$$

$$\text{Torque} = 795 \times {\cdot}14 = 111 \text{ lb. in.}$$

$$Z_{XX} = \frac{\pi}{32} bd^2 = {\cdot}219 \text{ in.}^3$$

$$Z_{YY} = \frac{\pi}{32} b^2d = {\cdot}452 \text{ in.}^3$$

Neglecting the torque, which is small,

$$f_{XX} = \frac{1880}{1{\cdot}71} + \frac{901}{{\cdot}219} = 1100 + 4120 = 5220 \text{ lb./in.}^2$$

$$f_{YY} = \frac{1880}{1{\cdot}71} + \frac{3260}{{\cdot}452} = 1100 + 7220 = 8320 \text{ lb./in.}^2$$

APPENDIX

Useful Data.

Acceleration due to gravity $(g) = 32{\cdot}2$ ft./sec.^{2}.

60 m.p.h. $= 88$ ft./sec.

1 in. $= 2{\cdot}54$ cm.

π radians $= 180$ deg.

1 radian $= 57{\cdot}3$ deg.

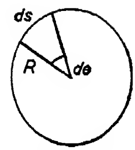

FIG 114

$$\frac{\text{arc}}{\text{radius}} = \text{radians} \quad \text{or} \quad \frac{ds}{R} = d\theta \text{ (Fig. 114).}$$

$\sin 30 = \frac{1}{2}$ $\quad$ $\sin 60 = \frac{\sqrt{3}}{2}$

$\cos 30 = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 60 = \frac{1}{2}$ $\quad$ Fig. 115.

$\tan 30 = \frac{1}{\sqrt{3}}$ $\quad$ $\tan 60 = \sqrt{3}$

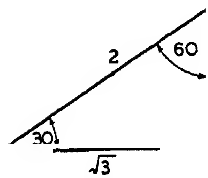

FIG. 115.

$\sin 90 = 1$ $\quad$ $\sin 45 = \frac{1}{\sqrt{2}}$

$\cos 90 = 0$ $\quad$ $\cos 45 = \frac{\ }{\sqrt{2}}$

$\log_{10} 1 = 0$ $\quad$ $\log_e x = 2{\cdot}3025 \log_{10} x$

$\log_{10} 10 = 1{\cdot}0$ | $\log_{10} x = {\cdot}4343 \log_e x$

$e = 2{\cdot}7183$

Density of air $(\rho) = {\cdot}002378$ slugs per cubic ft.

Fundamental Identity.

$$\left.\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= +\cos\theta \\ \tan(-\theta) &= -\tan\theta \end{aligned}\right\} \underline{\cos + \text{ve}}$$

$$\left.\begin{array}{l}\sin(180+\theta) = -\sin\theta \\ \cos(180+\theta) = -\cos\theta \\ \tan(180+\theta) = +\tan\theta\end{array}\right\} \tan + \text{ve}$$

$$\left.\begin{array}{l}\sin(180-\theta) = +\sin\theta \\ \cos(180-\theta) = -\cos\theta \\ \tan(180-\theta) = -\tan\theta\end{array}\right\} \underline{\sin + \text{ve}}$$

$$\left.\begin{array}{l}\sin(90+\theta) = +\cos\theta \\ \cos(90+\theta) = -\sin\theta \\ \tan(90+\theta) = -\cot\theta\end{array}\right\} \underline{\cos + \text{ve}}$$

$$\begin{array}{l}\sin(90-\theta) = \cos\theta \\ \cos(90-\theta) = \sin\theta \\ \tan(90-\theta) = \cot\theta\end{array} \qquad \text{all} + \text{ve}$$

$$\begin{array}{l}\cos(A+B) = \cos A\cos B - \sin A\sin B \\ \cos(A-B) = \cos A\cos B + \sin A\sin B \\ \sin(A+B) = \sin A\cos B + \cos A\sin B \\ \sin(A-B) = \sin A\cos B - \cos A\sin B\end{array}$$

whence

$$\begin{array}{l}2\cos A\cos B = \cos(A+B) + \cos(A-B) \\ 2\sin A\sin B = \cos(A-B) - \cos(A+B) \\ 2\sin A\cos B = \sin(A+B) + \sin(A-B) \\ 2\cos A\sin B = \sin(A+B) - \sin(A-B)\end{array}$$

and

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

Differentials.

y.	$\frac{dy}{dx}$.
x^n	nx^{n-1}
$\frac{1}{x}=x^{-1}$	$-x^{-2}$
$\sin x$	$\cos x$
$\sin ax$	$a \cos ax$
$\cos x$	$-\sin x$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
e^x	e^x
e^{ax}	ae^{ax}
$\log_e x$	$\frac{1}{x}$
Product.	
uv	$\frac{vdu}{dx}+\frac{udv}{dx}$
e.g. $\sin x \cos x$	$\cos x \cos x - \sin x \sin x$ $\cos^2 x - \sin^2 x$
Quotient.	
$\frac{u}{v}$	$\frac{1}{v^2}\left(\frac{vdu}{dx}-\frac{udv}{dx}\right)$
e.g. $\frac{\sin x}{\cos x}$	$\frac{1}{\cos^2 x}(\cos^2 x - \sin^2 x)$

Integrals.

y.	$\int ydx$
$\frac{1}{x}$	$\log_e x$
$\sin wx$	$-\frac{1}{w}\cos wx$
$\cos wx$	$\frac{1}{w}\sin wx$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$

Double and Triple Angles.

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$= \cos^2 A - \sin^2 A.$$

Also

$$2\cos^2 A = 1+\cos 2A$$

and

$$2\sin^2 A = 1-\cos 2A.$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\sin 3A = 3\sin A - 4\sin^3 A.$$

$$\sin^2\theta+\cos^2\theta = 1$$

$$\sec^2\theta = 1+\tan^2\theta$$

$$\operatorname{cosec}^2\theta = 1+\cot^2\theta.$$

Identity.

To prove that

$$A\sin\mu x + B\cos\mu x = C\cos(\mu x-\Sigma).$$

Let

$$m = A\sin\mu x + B\cos\mu x \quad\text{and}\quad \sqrt{A^2+B^2}=C.$$

Then, multiplying top and bottom by $\sqrt{A^2+B^2}$,

$$m = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\sin\mu x + \frac{B}{\sqrt{A^2+B^2}}\cos\mu x\right]$$

$$= \sqrt{A^2+B^2}\,[\sin\Sigma\sin\mu x+\cos\Sigma\cos\mu x]$$

from Fig. 116

$$= \sqrt{A^2+B^2}\cos(\mu x-\Sigma) = C\cos(\mu x-\Sigma).$$

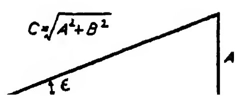

FIG. 116.—Diagram for proof of identity

Generalized Equation of Three Moments.

(See Air Publication 970, VI, 6.)

Proof that

$$f(a) = \frac{3}{2}\left[\frac{2a\operatorname{cosec}2a-1}{a^2}\right] = 2\phi(a)-\phi\left(\frac{a}{2}\right),$$

where

$$\phi(a) = \frac{3}{4}\left[\frac{1-2a\cot 2a}{a^2}\right]$$

so that

$$2\phi(a) = \frac{3}{2}\left[\frac{1-2a\cot 2a}{a^2}\right]$$

and

$$\left.\frac{\alpha}{2}\right)=\frac{3}{4}\left(\frac{1-\alpha\cot\alpha}{\frac{\alpha^2}{4}}\right)=\frac{3\,(1-\alpha\cot\alpha)}{\alpha^2}.$$

$$\text{R.H.S.}=2\phi(\alpha)-\phi\left(\frac{\alpha}{2}\right)=\frac{3}{2}\left[\frac{1-2\alpha\cot 2\alpha}{\alpha^2}\right]-3\left[\frac{1-\alpha\cot\alpha}{\alpha^2}\right]$$

$$=\frac{3}{2\alpha^2}\left[1-2\alpha\cot 2\alpha-2(1-\alpha\cot\alpha)\right]$$

$$=\frac{3}{2\alpha^2}\left[2\alpha\,(\cot\alpha-\cot 2\alpha)-1\right]$$

$$\text{L.H.S.}=\frac{3}{2\alpha^2}\left[2\alpha\operatorname{cosec}2\alpha-1\right].$$

We have to prove, therefore, that

$$(2\alpha\operatorname{cosec}2\alpha-1)\equiv 2\alpha(\cot\alpha-\cot 2\alpha)-1,$$

i.e.

$$\frac{2\alpha}{\sin 2\alpha}-1\equiv 2\alpha\left(\frac{\cos\alpha}{\sin\alpha}-\frac{\cos 2\alpha}{\sin 2\alpha}\right)-1$$

$$\equiv 2\alpha\left(\frac{2\cos^2\alpha-\cos^2\alpha+\sin^2\alpha}{\sin 2\alpha}\right)-1$$

$$\equiv 2\alpha\,.\,\frac{1}{\sin 2\alpha}-1.$$

Hence L.H S. = R.H.S.

Southwell's Formula.

Proof of alternative statement (see chapter on "Struts").

$$p_2=\frac{P}{A}+\frac{Peh\sec\alpha}{Ak^2}$$

$$=p\left[1+\frac{eh}{k^2}\sec\alpha\right]$$

$$=p\,[1+\lambda\sec\alpha],\qquad\text{where}\quad\lambda=\frac{eh}{k^2};$$

$$=p\left[1+\lambda\sec\frac{1}{2}\sqrt{\frac{P}{EI}}\right],\quad\text{where}\quad\alpha=\frac{1}{2}\sqrt{\frac{P}{EI}};$$

$$=p\left[1+\lambda\sec\frac{1}{2k}\sqrt{\frac{p}{E}}\right]$$

i.e. transposing,

$$p = \frac{p_2}{1 + \lambda \sec \dfrac{1}{2k}\sqrt{\dfrac{p}{E}}}$$

$$= \frac{p_2}{1 + \dfrac{eh}{k^2} \sec \dfrac{1}{2k}\sqrt{\dfrac{p}{E}}}.$$

Gauge Sizes.

S W.G	Size (in)	S W.G.	Size (in).
8	·160	18	048
10	128	20	036
12	·104	22	028
14	08	24	022
16	064	26	018
17	·056		

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